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F-SINGULARITIES UNDER GENERIC LINKAGE

LINQUAN MA, JANET PAGE, REBECCA R.G., WILLIAM TAYLOR, AND WENLIANG ZHANG

Dedicated to Prof. Craig Huneke on the occasion of his 65th birthday

ABSTRACT. Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a perfect field of positive characteristic. Let I be an equi-dimensional ideal in R and let J be a generic link of I in $S = R[u_{ij}]_{c \times r}$. We describe the parameter test submodule of S/J in terms of the test ideal of the pair (R, I) when a reduction of I is a complete intersection or almost complete intersection. As an application, we deduce a criterion for when S/J has F -rational singularities in these cases. We also compare the F -pure threshold of (R, I) and (S, J) .

1. INTRODUCTION

Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field of positive characteristic. Let $I = (f_1, \dots, f_r)$ be an equi-dimensional ideal in R of height c , where equi-dimensional means that all associated primes of I have the same height [Mat86]. We can define a regular sequence g_1, \dots, g_c in $S = R[u_{ij}]_{c \times r}$ via $g_i := u_{i1}f_1 + \dots + u_{ir}f_r$, where the u_{ij} are variables over S . Then $J = (g_1, \dots, g_c) : I$ is called a generic link of I in $S = R[u_{ij}]$. The study of generic linkage has attracted considerable attention and has been developed widely from both algebraic and geometric points of view [HU87], [HU88], [CU02], [EHU04], [Niu14].

In contrast to the quick and deep development of singularity theories in the past decades, much less has been known about the behaviors of singularities under generic linkage. A special case is a result of Chardin and Ulrich [CU02] which says that if R/I is a complete intersection and has rational (resp. F -rational) singularities, then a generic link S/J also has rational (resp. F -rational) singularities). This result in characteristic zero has been vastly extended in recent work of Niu [Niu14], which is our main motivation for this research.

Theorem 1.1 (Theorem 1.1 in [Niu14]). *Let J be a generic link of a reduced and equidimensional ideal I in $S = R[u_{ij}]$ and assume that the characteristic of k is 0. We have*

- (1) $\omega_{S/J}^{GR} \cong \mathcal{J}(R, I^c) \cdot (S/J)$, where $\omega_{S/J}^{GR}$ denotes the Grauert-Riemenschneider canonical sheaf of S/J and $\mathcal{J}(R, I^c)$ denotes the multiplier ideal of the pair (R, I^c) ,
- (2) $\text{lct}(S, J) \geq \text{lct}(R, I)$. In particular, if the pair (R, I^c) is log canonical, then the pair (S, J^c) is also log canonical.

This result gives a nice criterion for a generic link to have rational singularities in characteristic 0. It also has applications to bounding the Castelnuovo-Mumford regularity of projective varieties [Niu14, Corollary 1.2]. Since test ideals and F -pure thresholds are characteristic p analogues of multiplier ideals and log canonical thresholds (c.f. [BST15] and [HY03]), it is natural to ask whether analogues of Theorem 1.1 hold for test ideals and F -pure thresholds. Our main result is the following, which partially extends Theorem 1.1 to characteristic p and generalizes [CU02, Theorem 3.13] in characteristic p .

Theorem 1.2 (Theorem 3.3, Corollary 4.4). *Let J be a generic link of an equi-dimensional ideal I in $S = R[u_{ij}]$ and assume the characteristic of k is $p > 0$.*

- (1) *Suppose I is reduced and that a reduction of I is a complete intersection or an almost complete intersection. Then $\tau(\omega_{S/J}) \cong \tau(R, I^c) \cdot (S/J)$, where $\tau(\omega_{S/J})$ denotes the parameter test submodule and $\tau(R, I^c)$ denotes the test ideal of the pair (R, I^c) .*
- (2) *Suppose that a reduction of I is a complete intersection. Then $\text{fpt}_S(J) \geq \text{fpt}_R(I)$. In particular, if the pair (R, I^c) is F -pure, then the pair (S, J^c) is also F -pure.*

This paper is organized as follows: in Section 2 we recall and prove some basic result for F -singularities and test ideals; in Section 3 we give a description of the parameter test submodule of S/J in terms of the test

ideal of the pair (R, I) , when a reduction of I is a complete intersection or an almost complete intersection. This generalizes earlier results in [CU02]. In Section 4 we compare the F -pure threshold of the pairs (S, J) and (R, I) when a reduction of I is a complete intersection.

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2. F -SINGULARITIES AND TEST IDEALS

In this section we collect some basic definitions of F -singularities and test ideals and prove a characteristic $p > 0$ analogue of Ein's Lemma in [Niu14], which will be used in later sections.

Let R be a Noetherian commutative ring of characteristic $p > 0$. We will use $F_*^e R$ to denote the target of the e -th Frobenius endomorphism $F^e : R \xrightarrow{r \mapsto r^{p^e}} R$, i.e. $F_*^e R$ is an R -bimodule, which is the same as R as an abelian group and as a right R -module, that acquires its left R -module structure via the e -th Frobenius endomorphism $F^e : R \xrightarrow{r \mapsto r^{p^e}} R$. When R is reduced, we will use R^{1/p^e} to denote the ring whose elements are p^e -th roots of elements of R . Note that these notations (when R is reduced) $F_*^e R$ and R^{1/p^e} are used interchangeably in the literature; we will do so in this paper as well assuming no confusion will arise.

Remark 2.1. If R is a commutative ring essentially of finite type over a perfect field of characteristic $p > 0$, then R admits a canonical module denoted by ω_R . Applying $\text{Hom}_R(-, \omega_R)$ to the e -th Frobenius $R \rightarrow F_*^e R$ produces an R -linear map

$$\text{Hom}_R(F_*^e R, \omega_R) \rightarrow \text{Hom}_R(R, \omega_R) = \omega_R.$$

Moreover, we have $F_*^e \omega_R \cong \text{Hom}_R(F_*^e R, \omega_R)$ (see [BST15, Example 2.4] for more details). Hence we have a natural R -linear map:

$$\Phi_R^e : F_*^e \omega_R \cong \text{Hom}_R(F_*^e R, \omega_R) \rightarrow \text{Hom}_R(R, \omega_R) = \omega_R$$

called the trace map of the e -th Frobenius.

Example 2.2. When $R = k[x_1, \dots, x_n]$ is a polynomial ring over a perfect field k of characteristic $p > 0$, we can identify ω_R with R , and Φ_R^e can be identified with the usual trace Tr_R^e , that is:

$$\text{Tr}_R^e(F_*^e(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n})) = \begin{cases} x_1^{\frac{i_1 - (p^e - 1)}{p^e}} x_2^{\frac{i_2 - (p^e - 1)}{p^e}} \cdots x_n^{\frac{i_n - (p^e - 1)}{p^e}}, & \text{if } \frac{i_t - (p^e - 1)}{p^e} \in \mathbb{Z} \text{ for each } t \\ 0, & \text{otherwise} \end{cases}$$

In this case $\text{Hom}_R(F_*^e R, R)$ is a cyclic $F_*^e R$ -module generated by Tr_R^e . Furthermore, if f_1, \dots, f_c is a regular sequence in R and $T = R/(f_1, \dots, f_c)$, then we have ([Fed83, Corollary on page 465]¹)

$$\Phi_T^e(F_*^e(-)) = \text{image of } \text{Tr}_R^e(F_*^e(f_1^{p^e-1} \cdots f_c^{p^e-1} \cdot -)) \text{ in } T.$$

Remark 2.3. Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k of characteristic $p > 0$ and $A = R[y_1, \dots, y_m]$ be a polynomial ring over R . For each ideal I in R , it is well known and straightforward to check that

$$\text{Tr}_R^e(F_*^e(I))A = \text{Tr}_A^e(F_*^e(IA)).$$

Lemma 2.4. *Let $S \rightarrow R$ be a surjection of Noetherian commutative rings of characteristic p . Assume that both S and R admit canonical module ω_S and ω_R respectively and $\dim S = \dim R$. Then*

$$\Phi_R^e = \Phi_S^e|_{\omega_R}.$$

¹Fedder's result [Fed83, Corollary on page 465] assumes that the ring R is a Gorenstein local ring only to ensure that $\text{Hom}_R(F_*^e R, R) \cong F_*^e R$. In our case, $R = k[x_1, \dots, x_n]$ is a polynomial ring, so $\text{Hom}_R(F_*^e R, R)$ is clearly isomorphic to $F_*^e R$. Hence Fedder's result applies in our case.

Proof. Under our assumptions, we have $\omega_R = \text{Hom}_S(R, \omega_S)$ and the surjection $S \rightarrow R$ induces an inclusion $\omega_R = \text{Hom}_S(R, \omega_S) \hookrightarrow \omega_S$. Consider the following diagram

$$\begin{array}{ccccc} \text{Hom}_R(F_*^e R, \text{Hom}_S(R, \omega_S)) & \longrightarrow & \text{Hom}_R(R, \text{Hom}_S(R, \omega_S)) & \xrightarrow{\sim} & \text{Hom}_S(R, \omega_S) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Hom}_S(F_*^e S, \omega_S) & \longrightarrow & \text{Hom}_S(S, \omega_S) & \xrightarrow{\sim} & \omega_S \end{array}$$

Note that the top row (resp. the bottom row) induces the trace map Φ_R^e (resp. Φ_S^e). To prove our lemma, it suffices to prove

- (a) the vertical map on the left is an inclusion, and
- (b) the diagram commutes

To prove (a), note that the vertical map on the left can be refined further as

$$\begin{aligned} \text{Hom}_R(F_*^e R, \text{Hom}_S(R, \omega_S)) &= \text{Hom}_S(F_*^e R, \text{Hom}_S(R, \omega_S)) \\ &\hookrightarrow \text{Hom}_S(F_*^e S, \text{Hom}_S(R, \omega_S)) \text{ since } F_*^e S \twoheadrightarrow F_*^e R \\ &\hookrightarrow \text{Hom}_S(F_*^e S, \omega_S) \text{ since } \text{Hom}_S(R, \omega_S) \hookrightarrow \omega_S \end{aligned}$$

To prove (b), note that the commutativity follows directly from the commutativity of

$$\begin{array}{ccc} S & \longrightarrow & F_*^e S \\ \downarrow & & \downarrow \\ R & \longrightarrow & F_*^e R \end{array}$$

□

Definition 2.5 (*cf.* Definition 3.1 in [Har01] and Definition 2.33 in [BST15]). Let R be an F -finite Noetherian integral domain of characteristic p . The *parameter test submodule* $\tau(\omega_R)$ is the unique smallest nonzero submodule M of ω_R such that $\Phi_R(F_* M) \subseteq M$. R is called *F -rational* if R is Cohen-Macaulay and $\tau(\omega_R) = \omega_R$. Note that this is not the original definition of F -rationality, but is known to be equivalent [Smi97].

Definition 2.6 (*cf.* Definition 3.16 and Theorem 3.18 in [Sch11]). Let R be an F -finite Noetherian integral domain of characteristic p . Let $I \subseteq R$ be a nonzero ideal and $t \in \mathbb{Q}_{\geq 0}$. We define the *test ideal* $\tau(R, I^t)$, abbreviated $\tau(I^t)$, to be the unique smallest nonzero ideal $J \subseteq R$ such that $\phi(F_*^e(I^{\lceil t(p^e-1) \rceil} J)) \subseteq J$ for all $e > 0$ and all $\phi \in \text{Hom}_R(F_*^e R, R)$.

Definition 2.7 (*cf.* Definitions 1.3 and 2.1 and Proposition 2.2 in [TW04]). Let R be an F -finite, local, Noetherian, integral domain of characteristic p . Let $I \subset R$ be an ideal and $t \geq 0$ be a real number.

- (1) The pair (R, I^t) is *F -pure* if for all large $e \gg 0$, there exists an element $d \in I^{\lceil t(p^e-1) \rceil}$ such that $(F_*^e d)R \hookrightarrow F_*^e R$ splits as an R -module homomorphism.
- (2) The pair (R, I^t) is *strongly F -regular* if for every $c \neq 0$ there exists $e \geq 0$ and $d \in I^{\lceil tp^e \rceil}$ such that $F_*^e(cd)R \hookrightarrow F_*^e R$ splits as an R -module homomorphism.
- (3) The *F -pure threshold* $\text{fpt}_R(I)$ of (R, I) is $\sup\{s \in \mathbb{R}_{\geq 0} \mid \text{the pair } (R, I^s) \text{ is } F\text{-pure}\}$, and when R is strongly F -regular, we also have $\text{fpt}_R(I) = \sup\{s \in \mathbb{R}_{\geq 0} \mid \text{the pair } (R, I^s) \text{ is strongly } F\text{-regular}\}$.

Remark 2.8. Note that when R is local, (R, I^t) is strongly F -regular if and only if $\tau(I^t) = R$. Indeed, suppose (R, I^t) is strongly F -regular. Pick a nonzero element $c \in J$ and take $e \gg 0$ and $d \in I^{\lceil tp^e \rceil}$ satisfying the conditions of strong F -regularity for c , and let $\phi : F_*^e R \rightarrow R$ be a map such that $\phi(F_*^e(cd)) = 1$. Then

$$\phi(F_*^e(I^{\lceil t(p^e-1) \rceil} J)) \supseteq \phi(F_*^e(I^{\lceil tp^e \rceil} J)) = R,$$

and so $\tau(I^t) = R$.

On the other hand, if $\tau(I^t) = R$, $0 \neq c \in R$, and $a \in I^{\lceil t \rceil}$, then there exists $e \geq 0$ and $\phi : F_*^e R \rightarrow R$ such that $\phi(F_*^e(I^{\lceil t(p^e-1) \rceil} acR)) = R$. Let $b \in I^{\lceil t(p^e-1) \rceil}$ and $f \in R$ such that $\phi(F_*^e(c(abf))) = 1$. Then we are done once we note that $abf \in I^{\lceil t \rceil} I^{\lceil t(p^e-1) \rceil} \subseteq I^{\lceil tp^e \rceil}$.

We will need the following important description of test ideals:

Theorem 2.9 (cf. Proof of Theorem 3.18 in [Sch11]). *With the notations as in Definition 2.6, for any nonzero $a \in \tau(I^t)$, we have:*

$$\tau(I^t) = \sum_{e \geq 0} \sum_{\phi} \phi(F_*^e(aI^{\lceil t(p^e-1) \rceil}))$$

where the inner sum runs over all $\phi \in \text{Hom}_R(F_*^e R, R)$.

Remark 2.10. With the notations as in Definition 2.6, the following holds ([BSTZ10, 3.3])

$$(2.10.1) \quad \tau(I^t) = \sum_{e \geq 0} \sum_{\phi \in \text{Hom}_R(F_*^e R, R)} \phi(F_*^e(dI^{\lceil tp^e \rceil}))$$

where d is a big test element (which is just a nonzero element in $\tau(R) = \tau(R, I^0)$).

If $R = k[x_1, \dots, x_n]$ is a polynomial ring over a perfect field k of characteristic $p > 0$, then one can set $d = 1$ in (2.10.1) and $\text{Hom}_R(F_*^e R, R)$ is a cyclic $F_*^e R$ -module generated by Tr_R^e as discussed earlier. Hence by (2.10.1),

$$\begin{aligned} \tau(I^t) &= \sum_{e \geq 0} \sum_{\phi \in \text{Hom}_R(F_*^e R, R)} \phi(F_*^e(aI^{\lceil t(p^e-1) \rceil})) = \sum_{e \geq 0} \text{Tr}_R^e(F_*^e(aI^{\lceil t(p^e-1) \rceil})), \text{ for any } a \in \tau(I^t) \\ &= \sum_{e \geq 0} \sum_{\phi \in \text{Hom}_R(F_*^e R, R)} \phi(F_*^e(I^{\lceil tp^e \rceil})) = \sum_{e \geq 0} \text{Tr}_R^e(F_*^e(I^{\lceil tp^e \rceil})) \end{aligned}$$

Remark 2.11. With the notations as in Definition 2.5, one can show that if $R_{a'}$ is regular, then for every sufficiently large power a of a' , $\tau(\omega_R) = \sum_e \Phi_R^e(F_*^e(a \cdot \omega_R))$. This can be proved by a similar argument as [ST12, Lemma 3.6, Lemma 3.8] so we omit the details.

The following result from [ST12] will also be used. These results were originally proved in [HY03] and [HT04], and they hold as long as R is F -finite. We will only state the version of these results that we need.

Lemma 2.12 (cf. Theorem 6.9 in [ST12]). *Let R be an integral domain essentially of finite type over a perfect field of characteristic $p > 0$ and let $I, J \subseteq R$ be nonzero ideals and $t \in \mathbb{R}_{\geq 0}$.*

- (1) *If J is a reduction of I , then $\tau(I^t) = \tau(J^t)$.*
- (2) *If J is generated by r elements, then $\tau(J^r) = J\tau(J^{r-1})$.*

We are ready to prove the characteristic $p > 0$ analogue of Ein's Lemma in [Niu14]:

Lemma 2.13 (Ein's Lemma in characteristic $p > 0$). *Let R be an integral domain essentially of finite type over a perfect field of characteristic $p > 0$ and let $I \subseteq R$ be an equi-dimensional and unmixed ideal of codimension c . If $\tau(I^{c-1}) = R$, then $\tau(I^c) = I$. In particular, if R is strongly F -regular and (R, I^c) is F -pure, then $\tau(I^c) = I$.*

Proof. The lemma will follow from the following two inclusions:

$$(2.13.1) \quad \tau(I^c) \subseteq I.$$

$$(2.13.2) \quad I\tau(I^{t-1}) \subseteq \tau(I^t) \text{ for all } t \geq 1.$$

Indeed, if $\tau(I^{c-1}) = R$, then $I = I\tau(I^{c-1}) \subseteq \tau(I^c) \subseteq I$, and so we have equality throughout.

Proof of (2.13.1). Since inclusion is a local condition, we may assume that R is local with maximal ideal \mathfrak{m} . By replacing R by $R[x]_{\mathfrak{m}R[x]}$, we may assume that R has infinite residue field: it is straightforward to check that $\tau(I^c)R[x]_{\mathfrak{m}R[x]} = \tau((IR[x]_{\mathfrak{m}R[x]})^c)$. Now let \mathfrak{p} be a minimal prime of I . Since I is equi-dimensional, $\dim R_{\mathfrak{p}} = c$. Hence $IR_{\mathfrak{p}}$ has a reduction $J \subseteq IR_{\mathfrak{p}}$ generated by c elements. Therefore, since test ideals localize,

$$\tau(I^c)R_{\mathfrak{p}} = \tau((IR_{\mathfrak{p}})^c) = \tau(J^c) = J\tau(J^{c-1}) \subseteq J \subseteq IR_{\mathfrak{p}}.$$

Since every associated prime of I is minimal, this inclusion holds for all associated primes of I , hence it holds globally, i.e. $\tau(I^c) \subseteq I$. \square

Proof of (2.13.2). This should be well known to experts in the field; we opt to provide a proof here since we could not locate a proper reference. Let $t \in \mathbb{R}_{\geq 1}$, and pick $0 \neq a \in \tau(I^t)$. Then

$$\begin{aligned}
I\tau(I^{t-1}) &= I \sum_{e \geq 0} \sum_{\phi} \phi \left(F_*^e \left(aI^{\lceil (t-1)(p^e-1) \rceil} \right) \right) \\
&= \sum_{e \geq 0} \sum_{\phi} \phi \left(F_*^e \left(aI^{\lceil p^e \rceil} I^{\lceil (t-1)(p^e-1) \rceil} \right) \right) \\
&\subseteq \sum_{e \geq 0} \sum_{\phi} \phi \left(F_*^e \left(aI^{p^e} I^{\lceil (t-1)(p^e-1) \rceil} \right) \right) \\
&= \sum_{e \geq 0} \sum_{\phi} \phi \left(F_*^e \left(aI^{p^e + \lceil (t-1)(p^e-1) \rceil} \right) \right) \\
&\subseteq \sum_{e \geq 0} \sum_{\phi} \phi \left(F_*^e \left(aI^{\lceil t(p^e-1) \rceil} \right) \right) \\
&= \tau(I^t),
\end{aligned}$$

where the inner sum runs over all $\phi \in \text{Hom}_R(F_*^e R, R)$ and the last inclusion following from the fact that

$$p^e + \lceil (t-1)(p^e-1) \rceil = \lceil p^e + (t-1)(p^e-1) \rceil = \lceil t(p^e-1) + 1 \rceil > \lceil t(p^e-1) \rceil. \quad \square$$

For the last statement, if (R, I^c) is F -pure, then the F -pure threshold of I is at least c . Since the F -pure threshold is the supremum of those values t for which (R, I^t) is strongly F -regular when R is strongly F -regular [TW04, Proposition 2.2], we have that (R, I^{c-1}) is strongly F -regular. This means that $\tau(I^{c-1}) = R$ by Remark 2.8, and hence the first statement of the lemma tells us $\tau(I^c) = I$. \square

3. F -RATIONALITY UNDER GENERIC LINKAGE

In this section, we investigate how F -singularities (*e.g.* F -purity, F -rationality, etc) behave under a generic linkage. To this end, we will also consider the behaviors of test ideals under a generic linkage. We begin with recalling the definition of a generic link.

Definition 3.1. Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a perfect field of positive characteristic. Let I be an equi-dimensional and unmixed ideal of R of height c . Fix a generating set $\{f_1, \dots, f_r\}$ of I . Let (u_{ij}) , $1 \leq i \leq c$, $1 \leq j \leq r$, be a $c \times r$ matrix of variables. Consider c elements g_1, \dots, g_c in $S = R[u_{ij}]$ defined by

$$g_i := u_{i1}f_1 + u_{i2}f_2 + \dots + u_{ir}f_r$$

for $1 \leq i \leq c$. Then $J = (g_1, \dots, g_c) : (IS)$ is called the *first generic link* of I with respect to $\{f_1, \dots, f_r\}$ (we also call S/J the generic link of R/I with respect to $\{f_1, \dots, f_r\}$).

Remark 3.2. It is well known that under the above assumptions, if I is reduced, then IS and J are *geometrically linked*, i.e., $IS = (g_1, \dots, g_c) : J$ and $IS \cap J = (g_1, \dots, g_c)$. Moreover, J is actually a prime ideal of height c [HU85, 2.6].

The following theorem is our main technical result in this section.

Theorem 3.3. *With the notation as in Definition 3.1, assuming I is reduced, we have*

- (1) $\tau(\omega_{S/J}) \subseteq \tau(I^c) \cdot (S/J)$;
- (2) *If I has a minimal reduction generated by at most $c+1$ elements, then $\tau(\omega_{S/J}) \supseteq \tau(I^c) \cdot (S/J)$; hence $\tau(\omega_{S/J}) = \tau(I^c) \cdot (S/J)$ in this case.*

Our proof of Theorem 3.3(2) requires considering different sets of generators of I . A priori, a generic link (S, J) depends on the choice of generators. The following lemma guarantees that the statement in Theorem 3.3(2) is independent of the choice of generators of I . Its proof follows the same line as the proof of [HU87, Proposition 2.11].

Lemma 3.4. *Let Λ_1 and Λ_2 be two sets of generators of I and let (S_1, J_1) and (S_2, J_2) be generic links of I with respect to Λ_1 and Λ_2 respectively. Then $\tau(\omega_{S_1/J_1}) \supseteq \tau(I^c) \cdot (S_1/J_1)$ iff $\tau(\omega_{S_2/J_2}) \supseteq \tau(I^c) \cdot (S_2/J_2)$.*

Proof. By considering $\Lambda_1 \cup \Lambda_2$, we can assume that $\Lambda_1 \subseteq \Lambda_2$. By induction on the difference between the cardinality of Λ_1 and Λ_2 , we may assume that Λ_2 has one more element than Λ_1 , *i.e.* we may assume that $\Lambda_1 = \{f_1, \dots, f_r\}$ and $\Lambda_2 = \Lambda_1 \cup \{f_{r+1}\}$.

Denote the height of I by c . Let $\{u_{ij} \mid 1 \leq i \leq c, 1 \leq j \leq r+1\}$ be indeterminates over R . Set $S_1 = R[u_{ij}]_{1 \leq i \leq c, 1 \leq j \leq r}$ and $S_2 = R[u_{ij}]_{1 \leq i \leq c, 1 \leq j \leq r+1}$. For $i = 1, \dots, c$, set

$$g_i := u_{i1}f_1 + \dots + u_{ir}f_r$$

and

$$h_i := u_{i1}f_1 + \dots + u_{i,r+1}f_{r+1}.$$

Then $J_1 = ((g_1, \dots, g_c) :_S IS)$ is the first generic link of I with respect to Λ_1 and $J_2 = ((h_1, \dots, h_c) :_{S_2} IS_2)$ is the first generic link of I with respect to Λ_2 .

It is clear that $S_2 = S_1[u_{i,r+1}, \dots, u_{c,r+1}]$. Since $f_{r+1} \in I$, we must have that $f_{r+1} = \sum_{j=1}^r a_j f_j$ for some $a_j \in R$. Let $\varphi : S_2 \rightarrow S_2$ be the automorphism given by the linear change of variables

$$u_{ij} \mapsto u_{ij} + u_{i,r+1}a_j$$

for $1 \leq i \leq c$ and $1 \leq j \leq r$ and

$$u_{i,r+1} \mapsto u_{i,r+1}$$

for $1 \leq i \leq c$.

We claim that $\varphi(J_1 S_2) = J_2$ and we reason as follows. For $i = 1, \dots, c$, we have that

$$\varphi(g_i) = \sum_{j=1}^r (u_{ij} + u_{i,r+1}a_j)f_j = \sum_{j=1}^r u_{ij}f_j + u_{i,r+1} \sum_{j=1}^r a_j f_j = \sum_{j=1}^r u_{ij}f_j + u_{i,r+1}f_{r+1} = h_i.$$

Now since $S_1 \hookrightarrow S_2$ is a faithfully flat extension, we have that

$$J_1 S_2 = ((g_1, \dots, g_c) :_{S_1} IS_1) S_2 = ((g_1, \dots, g_c) S_2 :_{S_2} IS_2),$$

and hence

$$\varphi(J_1 S_2) = \varphi((g_1, \dots, g_c) S_2 :_{S_2} IS_2) = (\varphi(g_1, \dots, g_c) S_2 :_{S_2} \varphi(IS_2)) = ((h_1, \dots, h_c) :_{S_2} IS_2) = J_2.$$

Let S_2^φ denote the S_1 -algebra that is the same as S_2 as a ring and whose S_1 -module structure is induced by $S_1 \hookrightarrow S_2 \xrightarrow{\varphi} S_2$. Then we have shown that $J_1 \otimes_{S_1} S_2^\varphi = J_2$ and hence $S_1/J_1 \otimes_{S_1} S_2^\varphi = S_2/J_2$. Combining Remarks 2.3 and 2.11, one can check that

$$\tau(\omega_{S_1/J_1}) \otimes_{S_1} S_2^\varphi = \tau(\omega_{S_1/J_1} \otimes_{S_1} S_2^\varphi)$$

where the right hand side is precisely $\tau(\omega_{S_2/J_2})$. Our lemma follows immediately since S_2^φ is faithfully flat over S_1 . \square

The following lemma is also needed in the proof of Theorem 3.3.

Lemma 3.5. *Let c, r be positive integers such that $c = r$ or $c = r - 1$. Let $\beta = (\beta_1, \dots, \beta_r)$ be an element of \mathbb{N}^r , where \mathbb{N} is the set of non-negative integers. Assume $\sum_i \beta_i = c(p^e - 1)$. Then there exist c elements $\alpha_1, \dots, \alpha_c$ in \mathbb{N}^r such that:*

- (1) each α_i has at most two nonzero entries;
- (2) the sum of the entries of each α_i is $p^e - 1$;
- (3) $\beta_j = \sum_i \alpha_{ij}$, where $\alpha_i = (\alpha_{i1}, \dots, \alpha_{ir})$.

Proof. We will induce on r . If $c = r = 1$, then $\beta = (p^e - 1)$ and we let $\alpha_1 = \beta$. If $c = 1, r = 2$, we have $\beta = (\beta_1, \beta_2)$ where $\beta_1 + \beta_2 = p^e - 1$ and we can let $\alpha_1 = (\beta_1, \beta_2)$ and again (1)-(3) hold.

If $c = r$ and $\beta_1 = \dots = \beta_c = p^e - 1$, then we can set α_i to be the vector with $p^e - 1$ at i -th spot and 0 elsewhere. Otherwise, there must be a $\beta_i < p^e - 1$. Without loss of generality, we assume that $\beta_r < p^e - 1$.

We claim that $\beta_j \geq p^e - 1 - \beta_r$ for some j between 1 and $r - 1$, and we reason as follows. If $c = r$, then there must be a j such that $\beta_j > p^e - 1$, and hence $\beta_j \geq p^e - 1 - \beta_r$. Now assume that $c = r - 1$. Suppose $\beta_i < p^e - 1 - \beta_r$ for all $i \leq r - 1$, as then we would have:

$$\sum_{i=1}^r \beta_i < (r-1)(p^e - 1 - \beta_r) + \beta_r \leq (r-2)(p^e - 1 - \beta_r) + (p^e - 1) \leq (r-1)(p^e - 1) = c(p^e - 1)$$

which contradicts the assumption that $\sum_{i=1}^r \beta_i = c(p^e - 1)$. So, there is a j between 1 and $r - 1$ such that $\beta_j \geq p^e - 1 - \beta_r$.

Set $\alpha_c := (0, \dots, 0, p^e - 1 - \beta_r, 0, \dots, \beta_r)$ where $p^e - 1 - \beta_r$ appears in the j -th spot. Consider

$$(\beta_1, \dots, \beta_{j-1}, \beta_j - (p^e - 1 - \beta_r), \beta_{j+1}, \dots, \beta_{r-1}).$$

This is an element of \mathbb{N}^{r-1} such that the sum of its entries is $(c-1)(p^e - 1)$. By our induction hypotheses, there are $\gamma_1, \dots, \gamma_{c-1} \in \mathbb{N}^{r-1}$ that satisfy (1), (2), and (3). For $1 \leq i \leq c-1$, setting α_i be γ_i with a 0 added to the end completes the proof of our lemma. \square

Proof of Theorem 3.3. By Remark 3.2, J is a minimal prime of (g_1, \dots, g_c) . Hence once we identify

$$\omega_{S/J} = \text{Hom}_{S/(g_1, \dots, g_c)}(S/J, S/(g_1, \dots, g_c)) = ((g_1, \dots, g_c) : J) \cdot (S/J) = I \cdot (S/J),$$

we know from Lemma 2.4 that

$$\Phi_{S/J}^e(F_*^e(-)) = \Phi_{S/(g_1, \dots, g_c)}^e(F_*^e(-))|_{\omega_{S/J}} = \text{Tr}_S^e(F_*^e(g_1^{p^e-1} \dots g_c^{p^e-1} \cdot -))|_{I \cdot (S/J)}.$$

Next we notice that for every $1 \leq k \leq r$, $(S/J)_{f_k} \cong R_{f_k}[u_{ij} | j \neq k]$ is regular. Hence for $N \gg 0$, f_k^N is a test element for S/J . Thus by Remark 2.11, we have:

$$(3.5.1) \quad \tau(\omega_{S/J}) = \sum_{e \geq 0} \Phi_{S/J}^e(F_*^e(f_k^N \cdot \omega_{S/J})) = \sum_{e \geq 0} \text{Tr}_S^e(F_*^e(g_1^{p^e-1} \dots g_c^{p^e-1} \cdot f_k^N \cdot IS)) \cdot (S/J)$$

Since $f_k \in I$ and R is regular, by Remark 2.10, for $N \gg 0$ we also have:

$$(3.5.2) \quad \tau(I^c) \cdot (S/J) = \sum_{e \geq 0} \text{Tr}_R^e(F_*^e((f_1, \dots, f_r)^{c(p^e-1)} \cdot f_k^N \cdot R)) \cdot (S/J)$$

When we expand $g_1^{p^e-1} \dots g_c^{p^e-1}$, it is easy to see from (3.5.1) that $\tau(\omega_{S/J})$ can be generated by elements of the form

$$(3.5.3) \quad \text{Tr}_S^e \left(F_*^e \left(\binom{p^e-1}{\alpha_{11}, \dots, \alpha_{1r}} \dots \binom{p^e-1}{\alpha_{c1}, \dots, \alpha_{cr}} f_1^{\beta_1} f_2^{\beta_2} \dots f_r^{\beta_r} \prod_{\substack{1 \leq i \leq c \\ 1 \leq j \leq r}} u_{ij}^{\alpha_{ij}} \cdot f_k^N \cdot s \cdot \prod_{\substack{1 \leq i \leq c \\ 1 \leq j \leq r}} u_{ij}^{\gamma_{ij}} \right) \right)$$

where $0 \leq \alpha_{ij} \leq p^e - 1$, $\beta_j = \sum_{i=1}^c \alpha_{ij}$, $\sum_{j=1}^r \beta_j = c(p^e - 1)$ and $s \in I$. By definition of the trace map, this is equal to

$$\prod_{\substack{1 \leq i \leq c \\ 1 \leq j \leq r}} u_{ij}^{\frac{\alpha_{ij} + \gamma_{ij} - (p^e - 1)}{p^e}} \cdot \text{Tr}_R^e \left(F_*^e \left(\binom{p^e-1}{\alpha_{11}, \dots, \alpha_{1r}} \dots \binom{p^e-1}{\alpha_{c1}, \dots, \alpha_{cr}} f_1^{\beta_1} f_2^{\beta_2} \dots f_r^{\beta_r} \cdot f_k^N \cdot s \right) \right)$$

where $\frac{\alpha_{ij} + \gamma_{ij} - (p^e - 1)}{p^e}$ denotes 0 if $\alpha_{ij} + \gamma_{ij} \not\equiv -1 \pmod{p^e}$. But it is clear that this element is in $\tau(I^c) \cdot S$ by expression (3.5.2). This proves (1).

Next we prove (2). By Lemma 3.4 we can assume that $\tilde{I} = (f_1, \dots, f_{c+1})$ is a reduction of I (the case that I has a reduction generated by c elements is similar). Hence by the arguments above, we have that, for $1 \leq k \leq c$ and $N \gg 0$,

$$\tau(I^c) \cdot (S/J) = \tau(\tilde{I}^c) \cdot (S/J) = \sum_{e \geq 0} \text{Tr}_R^e(F_*^e((f_1, \dots, f_{c+1})^{c(p^e-1)} \cdot f_k^{N+1} \cdot R)) \cdot (S/J)$$

Given a generator $f_1^{\beta_1} \dots f_{c+1}^{\beta_{c+1}}$ of $(f_1, \dots, f_{c+1})^{c(p^e-1)}$, we can find $\alpha_1, \dots, \alpha_c \in \mathbb{N}^{c+1}$ satisfying the conclusion of Lemma 3.5. Then

$$\prod_{\substack{1 \leq i \leq c \\ 1 \leq j \leq c+1}} (u_{ij} f_j)^{\alpha_{ij}} = \prod_{\substack{1 \leq i \leq c \\ 1 \leq j \leq c+1}} u_{ij}^{\alpha_{ij}} \prod_{1 \leq j \leq c+1} f_j^{\beta_j}$$

appears with coefficient $\binom{p^e-1}{\alpha_{11}, \dots, \alpha_{1, c+1}} \dots \binom{p^e-1}{\alpha_{c1}, \dots, \alpha_{c, c+1}}$ in the product $g_1^{p^e-1} \dots g_c^{p^e-1}$. Because each multinomial coefficient $\binom{p^e-1}{\alpha_{i1}, \dots, \alpha_{i, c+1}} = \binom{p^e-1}{\alpha_{ij_i}}$ for some j_i by Lemma 3.5 (1)-(2), they are nonzero in k .

Each α_{ij} is less than p^e , so let

$$s' = \left(\prod_{\substack{1 \leq i \leq c \\ 1 \leq j \leq c+1}} u_{ij}^{p^e - 1 - \alpha_{ij}} \right) \left(\prod_{\substack{1 \leq i \leq c \\ c+2 \leq j \leq r}} u_{ij}^{p^e - 1} \right).$$

Then $\mathrm{Tr}_S^e(F_*^e(- \cdot s'))$ sends $\prod_{1 \leq l \leq n} x_l^{p^e - 1} \prod_{\substack{1 \leq i \leq c \\ 1 \leq j \leq c+1}} u_{ij}^{\alpha_{ij}}$ to 1 and all other basis elements to 0. Hence,

$$\begin{aligned} & \mathrm{Tr}_S^e(F_*^e(g_1^{p^e - 1} \cdots g_c^{p^e - 1} \cdot f_k^{N+1} \cdot s' \cdot R)) \\ &= \mathrm{Tr}_S^e \left(F_*^e \left(\binom{p^e - 1}{\alpha_{11}, \dots, \alpha_{1,c+1}} \cdots \binom{p^e - 1}{\alpha_{c1}, \dots, \alpha_{c,c+1}} \cdot \prod_{\substack{1 \leq i \leq c \\ 1 \leq j \leq r}} u_{ij}^{p^e - 1} \prod_{j=1}^{c+1} f_j^{\beta_j} \cdot f_k^{N+1} R \right) \right) \\ &= \mathrm{Tr}_R^e \left(F_*^e \left(\prod_{j=1}^{c+1} f_j^{\beta_j} \cdot f_k^{N+1} \cdot R \right) \right). \end{aligned}$$

In particular,

$$\mathrm{Tr}_R^e \left(F_*^e \left(\prod_{j=1}^{c+1} f_j^{\beta_j} \cdot f_k^{N+1} \cdot R \right) \right) \cdot (S/J) = \mathrm{Tr}_S^e(F_*^e(g_1^{p^e - 1} \cdots g_c^{p^e - 1} \cdot f_k^N \cdot f_k s' R)) \cdot (S/J) \subseteq \tau(\omega_{S/J})$$

for every generator $\prod_{j=1}^{c+1} f_j^{\beta_j}$ of $(f_1, \dots, f_{c+1})^{c(p^e - 1)}$, where the second inclusion follows from expression (3.5.1). Therefore we have

$$\begin{aligned} \tau(I^c) \cdot (S/J) &= \tau(\tilde{I}^c) \cdot (S/J) \\ &= \sum_{e \geq 0} \mathrm{Tr}_R^e(F_*^e((f_1, \dots, f_{c+1})^{c(p^e - 1)} \cdot f_k^{N+1} \cdot R)) \cdot (S/J) \\ &\subseteq \tau(\omega_{S/J}). \end{aligned} \quad \square$$

Remark 3.6. The proof of Theorem 3.3 (2) requires the minimal reduction be generated by at most $c + 1$ elements. If not, then we are not in the case of Lemma 3.5 and it may be the case that there are always at least three nonzero entries in some α_i . Consequently, multinomial coefficients must be taken into consideration.

Corollary 3.7. *With the notation as in Definition 3.1 and the assumptions as in Theorem 3.3 (2), $\tau(\omega_{S/J}) = \omega_{S/J}$ if and only if $\tau(I^c) = I$. In particular, S/J has F -rational singularities if and only if S/J is Cohen-Macaulay and $\tau(I^c) = I$.*

Proof. If $\tau(I^c) = I$, then Theorem 3.3 immediately implies $\tau(\omega_{S/J}) = \omega_{S/J}$.

Conversely, assume that $\tau(I^c) \neq I$ and $\tau(\omega_{S/J}) = \omega_{S/J}$. Since $\tau(I^c)$ is always contained in I by (2.13.1), at least one of the generators of I is not in $\tau(I^c)$, say f_1 . From Theorem 3.3, we can see that $\tau(I^c)S + J = IS + J$; hence $f_1 \in \tau(I^c)S + J$. Thus, there are elements $a \in \tau(I^c)S$ and $b \in J$ such that $f_1 = a + b$. (Note that $b \neq 0$.) Then we have $f_1 - a \in J$ which implies that $(f_1 - a)f_1 \in (g_1, \dots, g_c)$. However, this is impossible because of the degrees in the u_{ij} . This is a contradiction.

The last assertion is clear because S/J is F -rational if and only if S/J is Cohen-Macaulay and $\tau(\omega_{S/J}) = \omega_{S/J}$. \square

Corollary 3.8. *With the notation as in Definition 3.1 and the assumptions as in Theorem 3.3 (2), if the pair (R, I^c) is F -pure and R/I is Cohen-Macaulay, then S/J is F -rational. In particular, if R/I is an F -pure complete intersection, then S/J is F -rational.*

Proof. By Lemma 2.13, (R, I^c) is F -pure implies $\tau(I^c) = I$. The first statement thus follows from Corollary 3.7. Finally, it is well known that when R/I is an F -pure complete intersection, the pair (R, I^c) is F -pure. This follows from a Fedder type criterion ([Tak04, Lemma 3.9] and others). \square

We can recover [Niu14, Corollary 3.4] in the complete intersection and almost complete intersection cases.

Corollary 3.9. *Let $I = (f_1, \dots, f_r)$ be an ideal of $\mathbb{C}[x_1, \dots, x_n]$ and let c be the codimension of I . Let S and J be in Definition 3.1. Assume that $r \leq c + 1$. Then S/J has rational singularities if and only if S/J is Cohen-Macaulay and $\mathcal{S}(I^c) = I$, where $\mathcal{S}(I^c)$ is the multiplier ideal of I^c .*

Proof. By [Smi97] and [Har98], S/J has rational singularities if and only if its reduction $(S/J)_p$ is F -rational for all $p \gg 0$. It is easy to see that, for $p \gg 0$, the reduction J_p of J is a generic link of the reduction I_p of I . Hence, S/J has rational singularities if and only if $(S/J)_p$ is Cohen-Macaulay and $\tau(I_p^c) = I_p$ for $p \gg 0$ by Corollary 3.7. On the other hand, it was proved in [HY03] that $\mathcal{S}(I^c)_p = \tau(I_p^c)$ for all $p \gg 0$. Therefore, we have S/J has rational singularities if and only if $(S/J)_p$ is Cohen-Macaulay and $\mathcal{S}(I^c)_p = I_p$ for $p \gg 0$. This completes the proof. \square

4. BEHAVIOR OF F -PURE THRESHOLD UNDER GENERIC LINKAGE

In this section we investigate behaviors of F -pure thresholds under generic linkage. We begin with an easy lemma.

Lemma 4.1. *Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a perfect field of characteristic p and I be an equi-dimensional and unmixed ideal of R . Let Λ_1 and Λ_2 be 2 sets of generators of I and let (S_i, J_i) be the generic link with respect to Λ_i ($i=1,2$). Then*

$$\text{fpt}_{S_1}(J_1) = \text{fpt}_{S_2}(J_2).$$

Proof. As in the proof of Lemma 3.4, we can assume that $\Lambda_1 = \{f_1, \dots, f_r\}$ and $\Lambda_2 = \{f_1, \dots, f_r, f_{r+1}\}$. Let φ and S_2^φ be the same as in the proof of Lemma 3.4. It is straightforward to check that

$$\tau(J_1^t) \otimes_{S_1} S_2^\varphi = \tau(J_2^t)$$

for each nonnegative real number t . Our lemma follows immediately. \square

Remark 4.2. Let $k \subseteq K$ be an extension of perfect fields and let $R = k[x_1, \dots, x_n]$ and $T = K[x_1, \dots, x_n]$. Since $\text{Hom}_R(R^{1/p^e}, R)$ and $\text{Hom}_T(T^{1/p^e}, T)$ are generated by the same projection, we have $\tau_R(I^t) = \tau_T((IT)^t)$ (c.f. [BMS08, Remark 2.18]).

Theorem 4.3. *Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a perfect field of characteristic p and I be an equi-dimensional and unmixed ideal of height c in R . Assume that $I = (f_1, \dots, f_s)$ and that I has a reduction \tilde{I} generated by r elements. Let $S = R[u_{ij}]_{1 \leq i \leq c, 1 \leq j \leq s}$ be a polynomial ring over R . For $1 \leq i \leq c$, let*

$$g_i = u_{i1}f_1 + u_{i2}f_2 + \dots + u_{is}f_s.$$

Then $\text{fpt}_S(g_1, \dots, g_c) \geq \frac{c}{r} \text{fpt}_R(I)$.

Proof. By Lemma 4.1, we can add the generators of \tilde{I} to those of I and then assume that $\tilde{I} = (f_1, \dots, f_r)$. Since \tilde{I} is a reduction of I , it follows from [TW04, Proposition 2.2(6)] that $\text{fpt}_R(I) = \text{fpt}_R(\tilde{I})$. Hence it suffices to show that $\tau_R(\tilde{I}^t) = R$ implies $\tau_S((g_1, \dots, g_c)^{\frac{ct}{r}}) = S$ for any positive real number $t < c$. To this end, assume that $\tau_R(\tilde{I}^t) = R$. By Remark 4.2, we may assume that k is algebraically closed.

We wish to show that $\tau_S((g_1, \dots, g_c)^{\frac{ct}{r}}) = S$. Suppose otherwise and we seek a contradiction. There is a maximal ideal \mathfrak{m} of S such that $\tau_S((g_1, \dots, g_c)^{\frac{ct}{r}}) \subseteq \mathfrak{m}$. Since k is algebraically closed, we can write $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n, u_{11} - b_{11}, \dots, u_{cr} - b_{cr})$ for some $a_i, b_{ij} \in k$. Set $\mathfrak{n} = (x_1 - a_1, \dots, x_n - a_n)$. Since $\tau_R(\tilde{I}^t) = R$, there exist an integer e , an R -linear map $\phi \in \text{Hom}_R(R^{1/p^e}, R)$, and nonnegative integers $\alpha_1, \dots, \alpha_r$ with $\sum_i \alpha_i = \lceil tp^e \rceil$ such that $\phi(f_1^{\alpha_1/p^e} \dots f_r^{\alpha_r/p^e}) \notin \mathfrak{n}$.

At this point we show that each $f_j \in \mathfrak{n}$, and therefore $\alpha_j \leq p^e - 1$ for all j . Indeed, let $e \geq 1$ such that $p^e \geq c/(c-t)$ and let $\psi : S^{1/p^e} \rightarrow S$ send the basis element $u_{1j}^{(p^e-1)/p^e} u_{2j}^{(p^e-1)/p^e} \dots u_{cj}^{(p^e-1)/p^e}$ to 1 and all other basis elements $x_\ell^{\alpha_\ell/p^e} u_{ij}^{b_{ij}/p^e}$ to 0. Now $f_j^c g_1^{p^e-1} g_2^{p^e-1} \dots g_c^{p^e-1} \in (g_1, \dots, g_c)^{\lceil (ct/r)p^e \rceil}$, because $(ct/r)p^e \leq tp^e \leq c(p^e - 1)$. Therefore

$$\begin{aligned} f_j^c &= \psi(f_j^{c/p^e} u_{1j}^{(p^e-1)/p^e} u_{2j}^{(p^e-1)/p^e} \dots u_{cj}^{(p^e-1)/p^e} f_j^{c(p^e-1)/p^e}) \\ &= \psi(f_j^{c/p^e} g_1^{(p^e-1)/p^e} g_2^{(p^e-1)/p^e} \dots g_c^{(p^e-1)/p^e}) \\ &\subseteq \psi(((g_1, \dots, g_c)^{\lceil (ct/r)p^e \rceil})^{1/p^e}) \subseteq \mathfrak{m} \end{aligned}$$

by our choice of ψ and the assumption that $\tau_S((g_1, \dots, g_c)^{\frac{ct}{r}}) \subseteq \mathfrak{m}$. It follows that $f_j \in \mathfrak{m} \cap R = \mathfrak{n}$.

Without loss of generality, we may assume that $p^e - 1 \geq \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_r$. Consequently,

$$\alpha_1 + \dots + \alpha_c \geq \left\lceil \frac{c}{r}(\alpha_1 + \dots + \alpha_r) \right\rceil = \left\lceil \frac{c}{r} \lceil tp^e \rceil \right\rceil \geq \left\lceil \frac{c}{r} tp^e \right\rceil$$

Let $\phi_{\underline{\alpha}} = \phi(f_{c+1}^{\alpha_{c+1}/p^e} \dots f_r^{\alpha_r/p^e} \cdot -)$, i.e. pre-multiplication by $f_{c+1}^{\alpha_{c+1}/p^e} \dots f_r^{\alpha_r/p^e}$ followed by the application of ϕ . It is clear that $\phi_{\underline{\alpha}} : R^{1/p^e} \rightarrow R$ is an R -linear map and that $\phi_{\underline{\alpha}}(f_1^{\alpha_1/p^e} \dots f_c^{\alpha_c/p^e}) \notin \mathfrak{n}$. We can extend $\phi_{\underline{\alpha}}$ to an S -linear map $\psi_{\underline{\alpha}} : R^{1/p^e}[u_{ij}] \rightarrow S = R[u_{ij}]$ that sends each u_{ij} to itself and restricts to $\phi_{\underline{\alpha}}$ on R^{1/p^e} .

It is clear that $S^{1/p^e} = R^{1/p^e}[u_{ij}^{1/p^e}]$ is a free $R^{1/p^e}[u_{ij}]$ -module with a basis $\{\prod_{0 \leq b_{ij} \leq p^e - 1} u_{ij}^{b_{ij}/p^e}\}$. Let $\pi_{\underline{\alpha}} : R^{1/p^e}[u_{ij}^{1/p^e}] \rightarrow R^{1/p^e}[u_{ij}]$ be the projection that sends $u_{11}^{\alpha_1/p^e} \dots u_{cc}^{\alpha_c/p^e}$ to 1 and all other basis elements to 0.

Let $\theta_{\underline{\alpha}}$ be the composition of $S^{1/p^e} \xrightarrow{\pi_{\underline{\alpha}}} R^{1/p^e}[u_{ij}] \xrightarrow{\psi_{\underline{\alpha}}} S$. It is clear that $\theta_{\underline{\alpha}}$ is S -linear. By the construction of $\pi_{\underline{\alpha}}$, it is straightforward to check that

$$\theta_{\underline{\alpha}}(g_1^{\alpha_1/p^e} \dots g_c^{\alpha_c/p^e}) = \theta_{\underline{\alpha}}((u_{11}f_1)^{\alpha_1/p^e} \dots (u_{cc}f_c)^{\alpha_c/p^e}) = \phi(f_1^{\alpha_1/p^e} \dots f_r^{\alpha_r/p^e}).$$

Since $\phi(f_1^{\alpha_1/p^e} \dots f_r^{\alpha_r/p^e})$ is in R but not in $\mathfrak{n} = (x_1 - a_1, \dots, x_n - a_n)$, we must have

$$\phi(f_1^{\alpha_1/p^e} \dots f_r^{\alpha_r/p^e}) \notin \mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n, u_{11} - b_{11}, \dots, u_{cr} - b_{cr}),$$

a contradiction to the assumption that $\tau_S((g_1, \dots, g_c)^{\frac{ct}{r}}) \subseteq \mathfrak{m}$ (note that $g_1^{\alpha_1} \dots g_c^{\alpha_c} \in (g_1, \dots, g_c)^{\lceil \frac{ct}{r} p^e \rceil}$). \square

We have some immediate corollaries.

Corollary 4.4. *Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a perfect field of characteristic p and I be an equi-dimensional and unmixed ideal of height c in R . Let J be a generic link of I in $S = R[u_{ij}]$. The following hold:*

- (1) *If I has a reduction generated by r elements, then $\text{fpt}_S(J) \geq \frac{c}{r} \text{fpt}_R(I)$.*
- (2) *If I has a reduction generated by c elements, in particular if I is a complete intersection, then $\text{fpt}_S(J) \geq \text{fpt}_R(I)$.*
- (3) *$\text{fpt}_S(J) \geq \frac{c}{n} \text{fpt}_R(I)$ (note $n = \dim(R)$).*

Proof. To prove (1), note that since $(g_1, \dots, g_c) \subseteq J$, we have $\text{fpt}_S(J) \geq \text{fpt}_S(g_1, \dots, g_c)$. Theorem 4.3 then completes the proof.

(2) is an immediate consequence of (1).

(3) By Remark 4.2, passing to the algebraic closure of k doesn't affect $\text{fpt}_R(I)$ and $\text{fpt}_S(J)$. Hence we can assume that k is algebraically closed and hence is infinite. [Lyu86, Theorem] asserts that each ideal I admits a reduction generated by n elements. We are done by (1). \square

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