Associative Structures Based Upon a Categorical Braiding

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ASSOCIATIVE STRUCTURES BASED UPON A CATEGORICAL BRAIDING

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Abstract. It is well known that the existence of a braiding in a monoidal category $\mathcal{V}$ allows many structures to be built upon that foundation. These include a monoidal 2-category $\mathcal{V}$-Cat of enriched categories and functors over $\mathcal{V}$, a monoidal bicategory $\mathcal{V}$-Mod of enriched categories and modules, a category of operads in $\mathcal{V}$ and a 2-fold monoidal category structure on $\mathcal{V}$. We will begin by focusing our exposition on the first and last in this list due to their ability to shed light on a new question. We ask, given a braiding on $\mathcal{V}$, what non-equal structures of a given kind in the list exist which are based upon the braiding. For instance, what non-equal monoidal structures are available on $\mathcal{V}$-Cat, or what non-equal operad structures are available which base their associative structure on the braiding in $\mathcal{V}$. All these examples are treated in one paper since they all require the same properties of the underlying braids of a transformation $\eta : (A \otimes B) \otimes (C \otimes D) \to (A \otimes C) \otimes (B \otimes D)$. The existence of duals in $\mathcal{V}$-Cat will give us an indication of where to look for the alternative underlying braids that result in an infinite family of associative structures. The external and internal associativity diagrams in the axioms of a 2-fold monoidal category will provide us with several obstructions that can prevent a braid from underlying an associative structure.

Contents

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1. Introduction

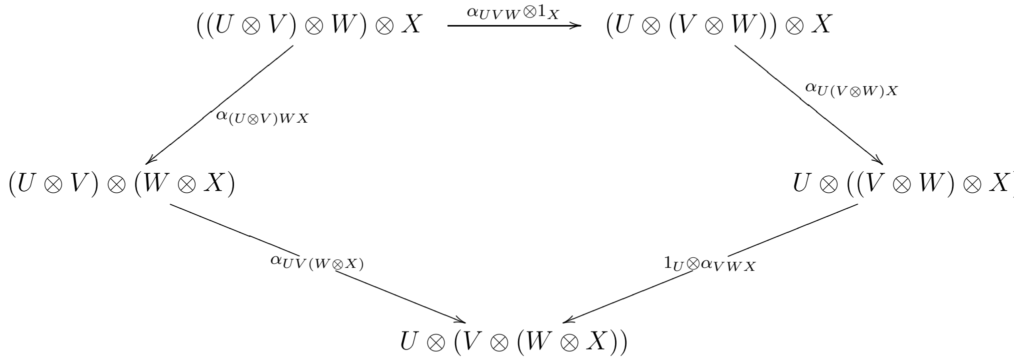
This paper begins with a review of the definition of the category of categories enriched over a monoidal category and then looks in detail at the additional structure that is available when the base category is braided. This structure has been studied in [Joyal and Street, 1993], where it was first noted that $\mathcal{V}$-Cat is braided only if $\mathcal{V}$ is symmetric. We repeat some of their results for exposition. The first new result here is the proof of existence of a large family of monoidal structures on $\mathcal{V}$-Cat based upon the left and right opposites of enriched categories. This introduces the true theme of the paper, which is the program for characterization of what we call associative braids. These are braids in B_{2n} that obey unit requirements and satisfy two equations in B_{3n}. In this paper we deal mainly with the case $n = 2$ which has the most implications for current category theory. Also in Section 2 there is an extension of a result in [Joyal and Street, 1993] about the nonexistence of a braiding on $\mathcal{V}$-Cat based upon the braiding of $\mathcal{V}$ for any monoidal structure based on the braiding of $\mathcal{V}$. The 3rd section reviews the axioms of a 2-fold monoidal category and the results that describe when a 2-fold monoidal structure gives rise to a braiding, and vice versa. The new contributions here consist of examples of obstructions to associativity that can easily be detected in a candidate for the interchange transformation built out of instances of a braiding. Finally we apply the new result about monoidal structures on $\mathcal{V}$-Cat directly to the existence of alternate 2-fold monoidal structures on $\mathcal{V}$, and apply the obstruction theory on 2-fold monoidalness to the viability of candidates for alternate monoidal structures on $\mathcal{V}$-Cat. In the final section we conclude with the application of the earlier results to operad theory.

2. Categories enriched over a braided monoidal category.

First we briefly review the definition of a category enriched over a monoidal category $\mathcal{V}$. Enriched functors and enriched natural transformations make the collection of enriched categories into a 2-category $\mathcal{V}$-Cat. The definitions and proofs can be found in more or less detail in [Kelly, 1982] and [Eilenberg and Kelly, 1965] and of course in [Mac Lane, 1998].

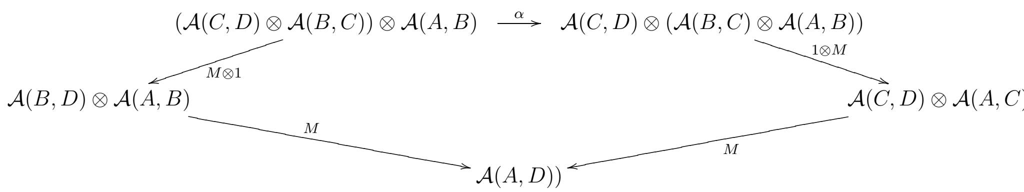
2.1. Definition. *For our purposes a monoidal category is a category $\mathcal{V}$ together with a functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and an object I such that*

1. *$\otimes$ is associative up to the coherent natural transformations α. The coherence axiom is given by the commuting pentagon*

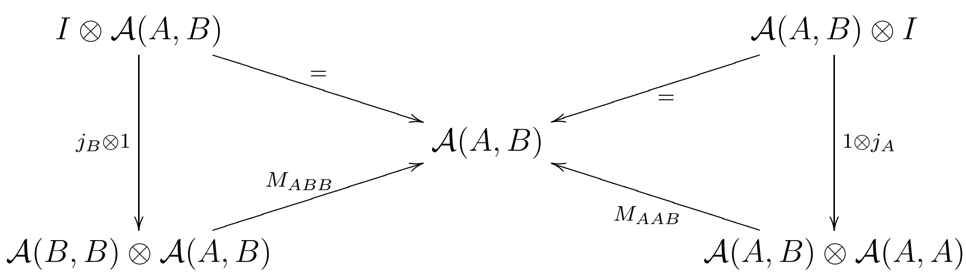


2. I is a strict 2-sided unit for $\otimes$.

2.2. Definition. *A (small) $\mathcal{V}$ -Category $\mathcal{A}$ is a set $|\mathcal{A}|$ of objects, a hom-object $\mathcal{A}(A, B) \in |\mathcal{V}|$ for each pair of objects of $\mathcal{A}$, a family of composition morphisms $M_{ABC} : \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \to \mathcal{A}(A, C)$ for each triple of objects, and an identity element $j_A : I \to \mathcal{A}(A, A)$ for each object. The composition morphisms are subject to the associativity axiom which states that the following pentagon commutes*



and to the unit axioms which state that both the triangles in the following diagram commute



In general a $\mathcal{V}$-category is directly analogous to an (ordinary) category enriched over **Set**. If $\mathcal{V} = \textbf{Set}$ then these diagrams are the usual category axioms. Basically, composition of morphisms is replaced by tensoring and the resulting diagrams are required to commute. The next two definitions exhibit this principle and are important since they give us the setting in which to construct a category of $\mathcal{V}$-categories.

2.3. Definition. *For $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, a $\mathcal{V}$-functor $T : \mathcal{A} \to \mathcal{B}$ is a function $T : |\mathcal{A}| \to |\mathcal{B}|$ and a family of morphisms $T_{AB} : \mathcal{A}(A, B) \to \mathcal{B}(TA, TB)$ in $\mathcal{V}$ indexed by pairs $A, B \in |\mathcal{A}|$. The usual rules for a functor that state $T(f \circ g) = Tf \circ Tg$ and $T1_A = 1_{TA}$ become in the enriched setting, respectively, the commuting diagrams*

$$\begin{CD} \mathcal{A}(B,C)\otimes\mathcal{A}(A,B) @>M>> \mathcal{A}(A,C) \\ @V{T\otimes T}VV @VV{T}V \\ \mathcal{B}(TB,TC)\otimes\mathcal{B}(TA,TB) @>M>> \mathcal{B}(TA,TC) \end{CD}$$

and

$$\begin{array}{ccc} & & \mathcal{A}(A,A) \\ & \overset{j_A}{\nearrow} & \downarrow{\scriptstyle T_{AA}} \\ I & & \\ & \underset{j_{TA}}{\searrow} & \\ & & \mathcal{B}(TA,TA) \end{array}$$

$\mathcal{V}$-functors can be composed to form a category called $\mathcal{V}$-Cat. This category is actually enriched over **Cat**, *the category of (small) categories with Cartesian product.*

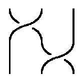
2.4. Definition. *For $\mathcal{V}$-functors $T, S : \mathcal{A} \to \mathcal{B}$ a $\mathcal{V}$-natural transformation $\alpha : T \to S : \mathcal{A} \to \mathcal{B}$ is an $|\mathcal{A}|$-indexed family of morphisms $\alpha_A : I \to \mathcal{B}(TA, SA)$ satisfying the $\mathcal{V}$-naturality condition expressed by the commutativity of*

$$\begin{array}{ccccc} & & I\otimes\mathcal{A}(A,B) \xrightarrow{\alpha_B\otimes T_{AB}} \mathcal{B}(TB,SB)\otimes\mathcal{B}(TA,TB) & & \\ & \overset{=}{\nearrow} & & \overset{M}{\searrow} & \\ \mathcal{A}(A,B) & & & & \mathcal{B}(TA,SB) \\ & \underset{=}{\searrow} & & \underset{M}{\nearrow} & \\ & & \mathcal{A}(A,B)\otimes I \xrightarrow[S_{AB}\otimes\alpha_A]{} \mathcal{B}(SA,SB)\otimes\mathcal{B}(TA,SA) & & \end{array}$$

For two $\mathcal{V}$-functors T, S to be equal is to say $TA = SA$ for all A and for the $\mathcal{V}$-natural isomorphism α between them to have components $\alpha_A = j_{TA}$. This latter implies equality of the hom-object morphisms: $T_{AB} = S_{AB}$ for all pairs of objects. The implication is seen by combining the second diagram in Definition 2.2 with all the diagrams in Definitions 2.3 and 2.4.

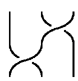
2.5. Definition. *A braiding for a monoidal category $\mathcal{V}$ is a family of natural isomorphisms $c_{XY} : X \otimes Y \to Y \otimes X$ such that the following diagrams commute. They are drawn next to their underlying braids.*

1.



$$\begin{array}{ccccc} & (X \otimes Y) \otimes Z & \xrightarrow{\alpha_{XYZ}} & X \otimes (Y \otimes Z) & \\ \swarrow^{c_{XY} \otimes 1} & & & & \searrow^{c_{X(Y \otimes Z)}} \\ (Y \otimes X) \otimes Z & & & & (Y \otimes Z) \otimes X \\ \searrow^{\alpha_{YXZ}} & & & & \swarrow_{\alpha_{YZX}} \\ & Y \otimes (X \otimes Z) & \xrightarrow{1 \otimes c_{XZ}} & Y \otimes (Z \otimes X) & \end{array}$$

2.



$$\begin{array}{ccccc} & X \otimes (Y \otimes Z) & \xrightarrow{\alpha^{-1}_{XYZ}} & (X \otimes Y) \otimes Z & \\ \swarrow^{1 \otimes c_{YZ}} & & & & \searrow^{c_{(X \otimes Y)Z}} \\ X \otimes (Z \otimes Y) & & & & Z \otimes (X \otimes Y) \\ \searrow^{\alpha^{-1}_{XZY}} & & & & \swarrow_{\alpha^{-1}_{ZXY}} \\ & (X \otimes Z) \otimes Y & \xrightarrow{c_{XZ} \otimes 1} & (Z \otimes X) \otimes Y & \end{array}$$

A braided category is a monoidal category with a chosen braiding.

Joyal and Street proved the coherence theorem for braided categories in [Joyal and Street, 1993], the immediate corollary of which is that in a free braided category generated by a set of objects, a diagram commutes if and only if all legs having the same source and target have the same underlying braid.

2.6. Definition. *A symmetry is a braiding such that the following diagram commutes*

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\quad 1 \quad} & X \otimes Y \\ & \searrow^{c_{XY}} \quad \nearrow^{c_{YX}} & \\ & Y \otimes X & \end{array}$$

In other words $c^{-1}_{XY} = c_{YX}$. A symmetric category is a monoidal category with a chosen symmetry.

If $\mathcal{V}$ is braided then we can define additional structure on $\mathcal{V}$-Cat. First there is a left opposite of a $\mathcal{V}$-category which has $|\mathcal{A}^{op}| = |\mathcal{A}|$ and $\mathcal{A}^{op}(A, A') = \mathcal{A}(A', A)$. The composition morphisms are given by

$$\begin{array}{c} \mathcal{A}^{op}(A', A'') \otimes \mathcal{A}^{op}(A, A') \\ \| \\ \mathcal{A}(A'', A') \otimes \mathcal{A}(A', A) \\ \downarrow c_{\mathcal{A}(A'',A')\otimes\mathcal{A}(A',A)} \\ \mathcal{A}(A', A) \otimes \mathcal{A}(A'', A') \\ \downarrow M_{AA'A''} \\ \mathcal{A}(A'', A) \\ \| \\ \mathcal{A}^{op}(A, A'') \end{array}$$

It is clear from this that $(\mathcal{A}^{op})^{op} \neq \mathcal{A}$. The pentagon diagram for the composition morphisms commutes since the braids underlying its legs are the two sides of the braid relation, also known as the Yang-Baxter equation. The right opposite denoted $\mathcal{A}^{po}$ is given by the same definition, but using c^{-1}. It is clear that $(\mathcal{A}^{po})^{op} = (\mathcal{A}^{op})^{po} = \mathcal{A}$. The second structure is a product for $\mathcal{V}$-Cat, that is, a 2-functor

$$\otimes^{(1)} : \mathcal{V}\text{-Cat} \times \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Cat}.$$

We will always denote the product(s) in $\mathcal{V}$-Cat with a superscript in parentheses that corresponds to the level of enrichment of the components of their domain. The product(s) in $\mathcal{V}$ should logically then have a superscript (0) but we have suppressed this for brevity and to agree with our sources. The product of two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ has $\left|\mathcal{A} \otimes^{(1)} \mathcal{B}\right| = |\mathcal{A}| \times |\mathcal{B}|$ and $(\mathcal{A} \otimes^{(1)} \mathcal{B})((A, B), (A', B')) = \mathcal{A}(A, A') \otimes \mathcal{B}(B, B')$.

The unit morphisms for the product $\mathcal{V}$-categories are the composites

$$I \cong I \otimes I \xrightarrow[j_A \otimes j_B]{} \mathcal{A}(A, A) \otimes \mathcal{B}(B, B)$$

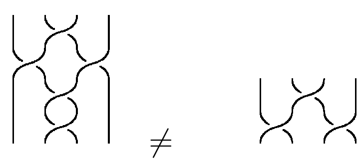
The composition morphisms

$$M_{(A,B)(A',B')(A'',B'')} : (\mathcal{A}\otimes^{(1)}\mathcal{B})((A', B'), (A'', B''))\otimes(\mathcal{A}\otimes^{(1)}\mathcal{B})((A, B), (A', B')) \to (\mathcal{A}\otimes^{(1)}\mathcal{B})((A, B), (A'', B''))$$

may be given by

$$
\begin{array}{c}
(\mathcal{A} \otimes^{(1)} \mathcal{B})((A', B'), (A'', B'')) \otimes (\mathcal{A} \otimes^{(1)} \mathcal{B})((A, B), (A', B')) \\
\Big\| \\
(\mathcal{A}(A', A'') \otimes \mathcal{B}(B', B'')) \otimes (\mathcal{A}(A, A') \otimes \mathcal{B}(B, B')) \\
\Big\downarrow {\scriptstyle (1 \otimes \alpha^{-1}) \circ \alpha} \\
\mathcal{A}(A', A'') \otimes ((\mathcal{B}(B', B'') \otimes \mathcal{A}(A, A')) \otimes \mathcal{B}(B, B')) \\
\Big\downarrow {\scriptstyle 1 \otimes (c_{\mathcal{B}(B',B'')\mathcal{A}(A,A')} \otimes 1)} \\
\mathcal{A}(A', A'') \otimes ((\mathcal{A}(A, A')) \otimes \mathcal{B}(B', B'')) \otimes \mathcal{B}(B, B')) \\
\Big\downarrow {\scriptstyle \alpha^{-1} \circ (1 \otimes \alpha)} \\
(\mathcal{A}(A', A'') \otimes \mathcal{A}(A, A')) \otimes (\mathcal{B}(B', B'')) \otimes \mathcal{B}(B, B')) \\
\Big\downarrow {\scriptstyle M_{AA'A''} \otimes M_{BB'B''}} \\
(\mathcal{A}(A, A'') \otimes \mathcal{B}(B, B'')) \\
\Big\| \\
(\mathcal{A} \otimes^{(1)} \mathcal{B})((A, B), (A'', B''))
\end{array}
$$

That $(\mathcal{A} \otimes^{(1)} \mathcal{B})^{op} \neq \mathcal{A}^{op} \otimes^{(1)} \mathcal{B}^{op}$ follows from the following braid inequality:

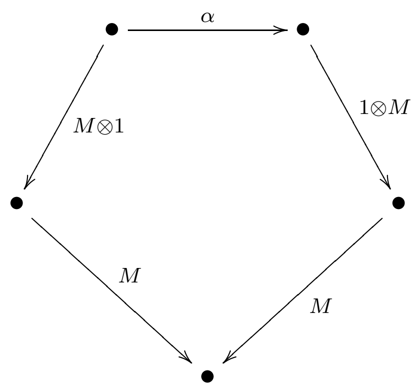


Now consider more carefully the morphisms of $\mathcal{V}$ that make up the composition morphism for a product enriched category, especially those that accomplish the interchange of the interior hom-objects. In the symmetric case, any other combination of instances of α and c with the same domain and range would be equal, due to coherence. In the merely braided case, there at first seems to be a much larger range of available choices. There is a canonical epimorphism $\sigma : B_n \to S_n$ of the braid group on n strands onto the permutation group. The permutation given by σ is that given by the strands of the braid on the n original positions. For instance on a canonical generator of B_n, σ_i, we have $\sigma(\sigma_i) = (i \;\; i+1)$. Candidates for multiplication would seem to be those defined using any braid $b \in B_4$ such that $\sigma(b) = (23)$. It is clear that the composition morphism would be defined as above, with a series of instances of α and c such that the underlying braid is b, followed in turn by $M_{AA'A''} \otimes M_{BB'B''}$ in order to complete the composition. That $M_{AA'A''} \otimes M_{BB'B''}$ will have the correct domain on which to operate is guaranteed by the permutation condition on b. For the unit $\mathcal{V}$-category $\mathcal{I}$ to be indeed a unit for one of the multiplications in question requires that in the underlying braid of the composition dropping either the first and third strand or the second and fourth strand leaves the identity on two strands. The unit axioms of the product categories are satisfied as long as dropping either the first two or the last two strands leaves again the identity on two strands. This is also due to the naturality of compositions of α and c and the unit axioms obeyed by $\mathcal{A}$ and $\mathcal{B}$. The remaining things to be checked are associativity of composition and functoriality of the associator.

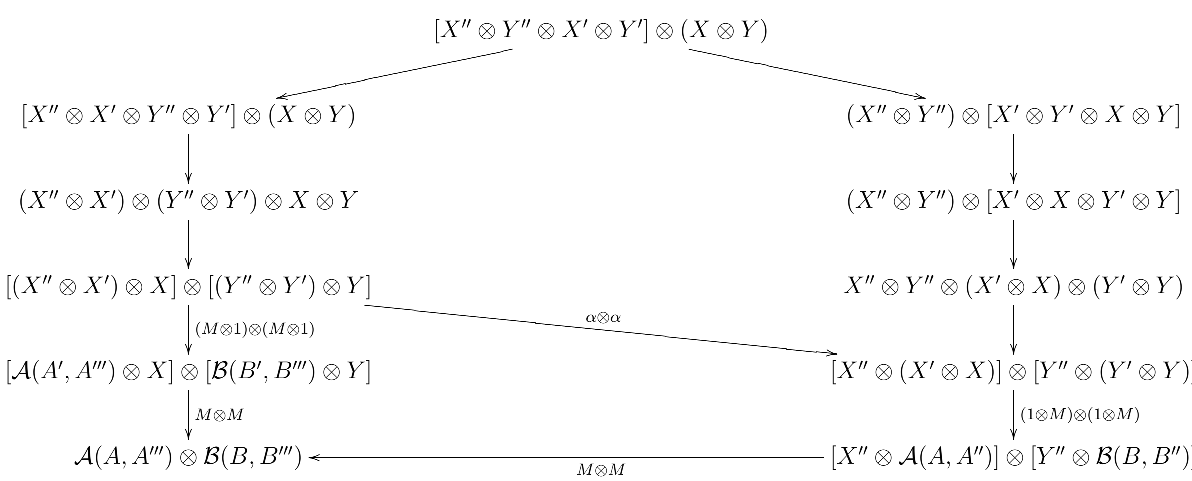
For the associativity axiom to hold the following diagram must commute, where the initial bullet represents

$$[(\mathcal{A}\otimes^{(1)}\mathcal{B})((A'',B''),(A''',B'''))\otimes(\mathcal{A}\otimes^{(1)}\mathcal{B})((A',B'),(A'',B''))]\otimes(\mathcal{A}\otimes^{(1)}\mathcal{B})((A,B),(A',B'))$$

and the last bullet represents $[\mathcal{A}\otimes^{(1)}\mathcal{B}]((A,B),(A''',B'''))$.



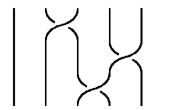
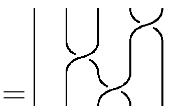
In $\mathcal{V}$ let $X = \mathcal{A}(A,A')$, $X' = \mathcal{A}(A',A'')$, $X'' = \mathcal{A}(A'',A''')$, $Y = \mathcal{B}(B,B')$, $Y' = \mathcal{B}(B',B'')$ and $Y'' = \mathcal{B}(B'',B''')$. The exterior of the following expanded diagram (where we leave out some parentheses for clarity and denote various composites of α and c by unlabeled arrows) is required to commute.

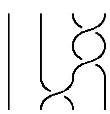
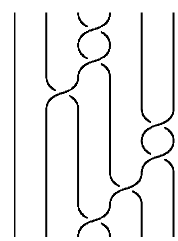
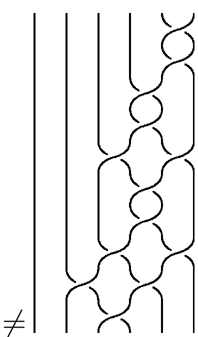


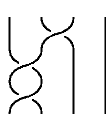
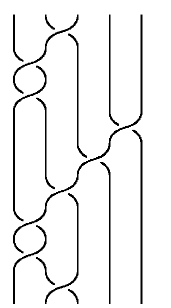
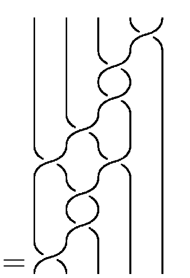
The bottom region commutes by the associativity axioms for $\mathcal{A}$ and $\mathcal{B}$. We are left needing to show that the underlying braids are equal for the two legs of the upper region.

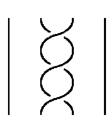
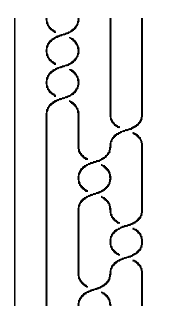
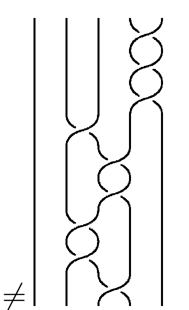
Again these basic nodes must be present regardless of the choice of braid by which the composition morphism is defined. Notice that the right and left legs have the following underlying braids in B_6 for some examples of various choices of b in B_4. We call the two derived braids in B_6 Lb and Rb respectively. Lb is algorithmically described as a copy of b on the first 4 strands followed by a copy of b on the 4 "strands" that result from pairing as the edges of two ribbons strands 1 and 2, and strands 3 and 4, along with the remaining two strands 5 and 6. Rb is similarly described, but the initial copy of b is on the last 4 strands, and the ribbon edge pairing is on the pairs 4,5 and 5,6. The first example for b is the one used in the original definition of $\otimes^{(1)}$ given above.

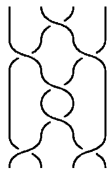
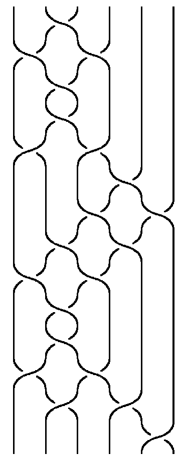
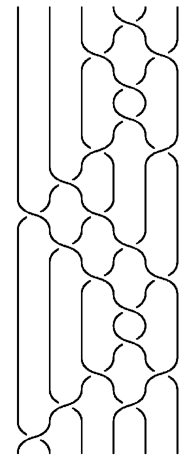
2.7. Example.

$b_{(1)} =$  Associativity follows from:  $=$ 

$b_{(2)} =$  Associativity does not follow since:  $\neq$ 

$b_{(3)} =$  Associativity follows from:  $=$ 

$b_{(4)} =$  Associativity does not follow since:  $\neq$ 

$b_{(5)} =$  Associativity does follow since:  = 

Before turning to check on functoriality of the associator, we note that $b_{(3)}$ is the braid underlying the composition morphism of the product category $(\mathcal{A}^{op})^{op} \otimes^{(1)} \mathcal{B}$ where the product is defined using $b_{(1)}$. This provides the hint that the two derived braids in B_6 that we are comparing above are equal because of the fact that the opposite of a $\mathcal{V}$-category is a valid $\mathcal{V}$-category. In fact we can describe sufficient conditions for Lb to be equivalent to Rb by describing the braids b that underlie the composition morphism of a product category given generally by $(((\mathcal{A}^{op})^{\cdots op} \otimes^{(1)} (\mathcal{B}^{op})^{\cdots op})^{op})^{\cdots op}$ where the number of *op* exponents is arbitrary in each position. Those braids are alternately described as lying in $H\sigma_2 K \subset B_4$ where H is the cyclic subgroup generated by the braid $\sigma_2\sigma_1\sigma_3\sigma_2$ and K is the subgroup generated by the two generators $\{\sigma_1, \sigma_3\}$. The latter subgroup K is isomorphic to $Z \times Z$. The first coordinate corresponds to the number of *op* exponents on $\mathcal{A}$ and the second component to the number of *op* exponents on $\mathcal{B}$. Negative integers correspond to the right opposites, *po*. The power of the element of H corresponds to the number of *op* exponents on the product of the two enriched categories, that is, the number of *op* exponents outside the parentheses. That $b \in H\sigma_2 K$ implies $Lb = Rb$ follows from the fact that the composition morphisms belonging to the opposite of a $\mathcal{V}$-category obey the pentagon axiom. An exercise of some value is to check consistency of the definitions by constructing an inductive proof of the implication based on braid group generators. This is not a necessary condition, but it may be when the additional requirement that $\sigma(b) = (23)$ is added. More work needs to be done to determine the necessary conditions and to study the structure and properties of the braids that meet these conditions.

Functoriality of the associator is necessary because here we need a 2-natural transformation $\alpha^{(1)}$. This means we have a family of $\mathcal{V}$-functors indexed by triples of $\mathcal{V}$-categories. On objects $\alpha^{(1)}_{\mathcal{ABC}}((A,B),C) = (A,(B,C))$. In order to guarantee that $\alpha^{(1)}$ obey the coherence pentagon for hom-object morphisms, we define it to be *based upon* α in $\mathcal{V}$. This means precisely that:

$$\alpha^{(1)}_{\mathcal{ABC}_{((A,B),C)((A',B'),C')}} : [(\mathcal{A}\otimes^{(1)}\mathcal{B})\otimes^{(1)}\mathcal{C}](((A,B),C)((A',B'),C')) \to [\mathcal{A}\otimes^{(1)}(\mathcal{B}\otimes^{(1)}\mathcal{C})]((A,(B,C))(A',(B',C')))$$

is equal to

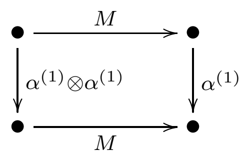
$$\alpha_{\mathcal{A}(A,A')\mathcal{B}(B,B')\mathcal{C}(C,C')} : (\mathcal{A}(A,A') \otimes \mathcal{B}(B,B')) \otimes \mathcal{C}(C,C') \to \mathcal{A}(A,A') \otimes (\mathcal{B}(B,B') \otimes \mathcal{C}(C,C')).$$

This definition guarantees that the $\alpha^{(1)}$ pentagons for objects and for hom-objects commute: the first trivially and the second by the fact that the α pentagon commutes in $\mathcal{V}$. We must also check for $\mathcal{V}$-functoriality. The unit axioms are trivial – we consider the more interesting axiom. The following diagram must commute, where the first bullet represents

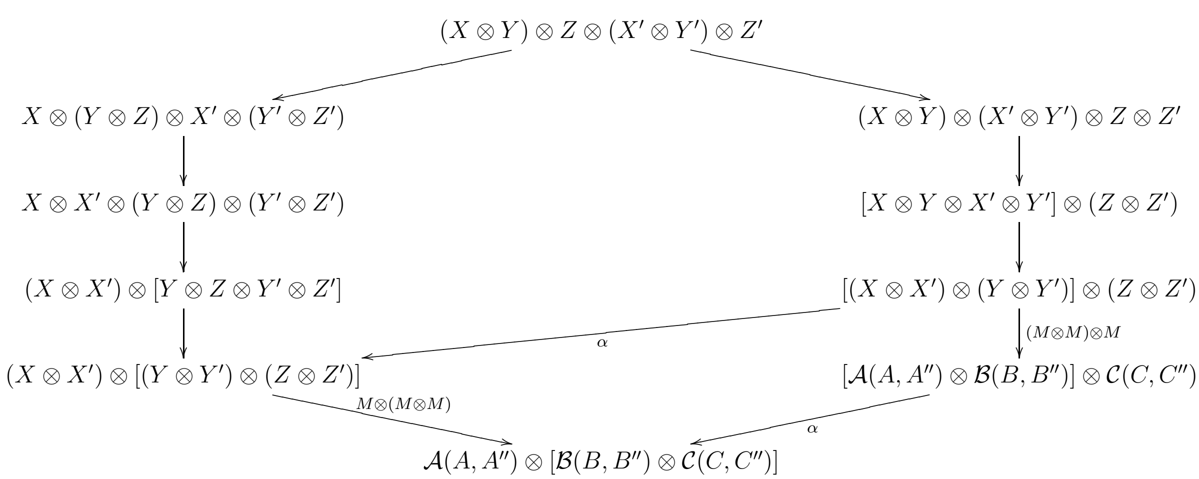
$$[(\mathcal{A}\otimes^{(1)}\mathcal{B})\otimes^{(1)}\mathcal{C}](((A',B'),C'),((A'',B''),C''))\otimes[(\mathcal{A}\otimes^{(1)}\mathcal{B})\otimes^{(1)}\mathcal{C}](((A,B),C),((A',B'),C'))$$

and the last bullet represents

$$[\mathcal{A} \otimes^{(1)} (\mathcal{B} \otimes^{(1)} \mathcal{C})]((A,(B,C)),(A'',(B'',C''))).$$

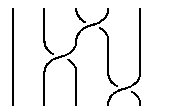
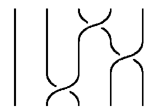


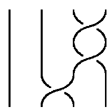
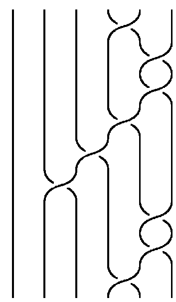
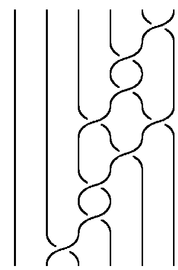
In $\mathcal{V}$ let $X = \mathcal{A}(A',A'')$, $Y = \mathcal{B}(B',B'')$, $Z = \mathcal{C}(C',C'')$, $X' = \mathcal{A}(A,A')$, $Y' = \mathcal{B}(B,B')$ and $Z' = \mathcal{C}(C,C')$ Then expanding the above diagram (where we leave out some parentheses for clarity and denote various composites of α and c by unlabeled arrows) we have

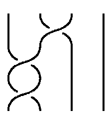
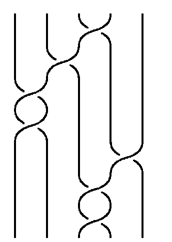
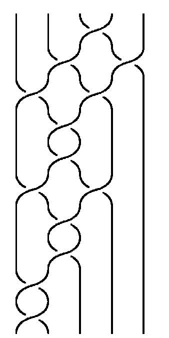


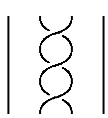
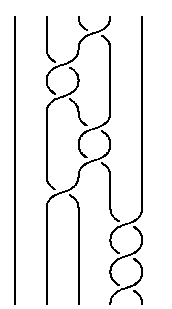
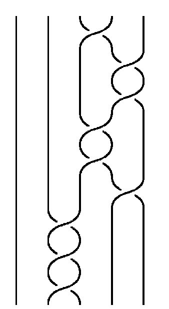
The bottom quadrilateral commutes by naturality of α. The top region must then commute for the diagram to commute. These basic nodes must be present regardless of the choice of braid by which the composition morphism is defined. Notice that the right and left legs have the following underlying braids for some examples of various choices of b. The two derived braids in B_6 we will refer to as $L'b$ and $R'b$. $L'b$ is formed from b by first pairing strands 2 and 3 , as well as strands 5 and 6 and performing b on the resulting four (groups of) strands. Then b is performed exactly on the last 4 strands. $R'b$ is derived in an analogous way as seen in the following examples. The first is the one used in the original definition of $\otimes^{(1)}$ given above.

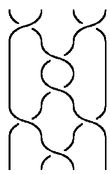
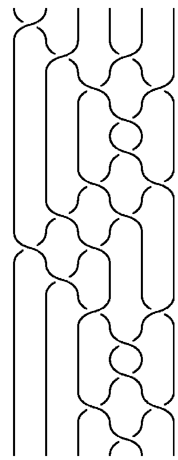
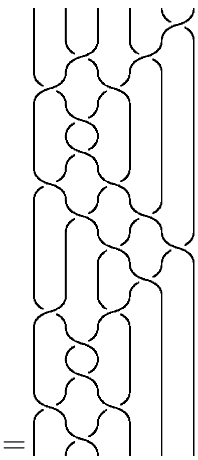
2.8. Example.

$b_{(1)} =$  Functoriality follows from:  $=$ 

$b_{(2)} =$  Functoriality follows from:  $=$ 

$b_{(3)} =$  Functoriality does not follow since:  $\neq$ 

$b_{(4)} =$  Functoriality does not follow since:  $\neq$ 

$b_{(5)} =$  Functoriality does follow since:  $=$ 

A comparison with the previous examples is of interest. Braids (2) and (3) are 180 degree rotations of each other. Notice that the second braid in the set of functoriality examples leads to an equality that is actually the same as for the third braid in the set of associativity examples. To see this the page must be rotated by 180 degrees. Similarly, the inequality preventing braid (2) from being associative is the 180 degree rotation of the inequality preventing braid (3) from being functorial. Braid (1) and braid (5) are each their own 180 degree rotation (the latter requires some deformation to make this evident), and the two braids proving each to be the underlying braid of an associative composition morphism are the same two that show each to underlie a functorial associator. Braid (4) is its own 180 degree rotation, and the two braids preventing it from being associative are the same two that obstruct it from being functorial. Thus there is a certain kind of duality between the requirements of associativity of the enriched composition and the functoriality of the associator. If we were considering a strictly associative monoidal category $\mathcal{V}$ then the condition of a functorial associator would become a condition of a well defined composition morphism. Including the coherent associator is somewhat more enlightening.

The question then is whether there are braids underlying the composition of a product of enriched categories besides the braids (1) and (5) above (and their inverses) that fulfill both obligations. The answer is yes.

2.9. Definition. *An candidate interchange braid on four strands is one for which the permutation associated to the braid is the middle exchange (23) and for which the unit conditions are satisfied, i.e deleting any one of the pairs of strands 1 and 2, 3 and 4, 1 and 3, or 2 and 4 results in the identity. An associative braid b is an interchanger candidate for which $Lb = Rb$ and $L'b = R'b$.*

To find associative braids we need only use the duality structure that exists on $\mathcal{V}$-Cat. By $\mathcal{A}^{op^n}$ is denoted the nth (left) opposite of $\mathcal{A}$. By $\otimes^{(1)}$ and $\otimes'^{(1)}$ we denote the standard multiplications defined respectively with braid (1) and its inverse.

2.10. Theorem. *The multiplication of enriched categories given by*

$$\mathcal{A} \otimes^{1(1)} \mathcal{B} = (\mathcal{A}^{op} \otimes'^{(1)} \mathcal{B}^{op})^{po}$$

is a valid monoidal product on $\mathcal{V}$-Cat. Furthermore, so are the multiplications

$$\mathcal{A} \otimes^{n(1)} \mathcal{B} = (\mathcal{A}^{op^n} \otimes'^{(1)} \mathcal{B}^{op^n})^{po^n}$$

as well as those with underlying braids that are the inverses of these, denoted $\mathcal{A} \otimes^{-n(1)} \mathcal{B}$.

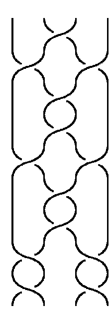
Proof: The first multiplication is mentioned alone since it has the underlying braid shown above as braid (5). Thus we have already demonstrated its fitness as a monoidal product. However this can be more efficiently shown just by noting that the category given by the product is certainly a valid enriched category, and that for three operands we have an associator from the isomorphism given by the following equation:

$$\begin{aligned}
&(((\mathcal{A}^{op} \otimes^{(1)} \mathcal{B}^{op})^{po})^{op} \otimes^{(1)} \mathcal{C}^{op})^{po} \\
&= ((\mathcal{A}^{op} \otimes^{(1)} \mathcal{B}^{op}) \otimes^{(1)} \mathcal{C}^{op})^{po} \\
&\cong (\mathcal{A}^{op} \otimes^{(1)} (\mathcal{B}^{op} \otimes^{(1)} \mathcal{C}^{op}))^{po} \\
&= (\mathcal{A}^{op} \otimes^{(1)} ((\mathcal{B}^{op} \otimes^{(1)} \mathcal{C}^{op})^{po})^{op})^{po}
\end{aligned}$$

The associator implicit in the isomorphism here is the same as the one defined above as based upon α in $\mathcal{V}$, since the object sets of the domain and range are identical and since:

$$[(\mathcal{A}^{op} \otimes^{(1)} \mathcal{B}^{op})^{po}]((A, B)(A', B')) = [\mathcal{A} \otimes^{(1)} \mathcal{B}]((A, B)(A', B'))$$

It is clear that this can be repeated with all the opposites and co-opposites raised to the nth degree. Recall that the unit $\mathcal{V}$-category $\mathcal{I}$ has only one object 0 and $\mathcal{I}(0, 0) = I$ the unit in $\mathcal{V}$. That this is indeed a unit for the multiplications in question follows from the fact that in any of the underlying braids dropping either the first and third strand or the second and fourth strand leaves the identity on two strands. The unit axioms of the product categories are satisfied due to the fact that dropping either the first two or the last two strands leaves again the identity on two strands. These geometrical facts can be observed by inspecting examples, such as the following braid that underlies $\otimes^{-2(1)}$:



Thus we have that, using any of the above multiplications including the standard ones $\otimes^{(1)}$ and $\otimes'^{(1)}$ defined respectively with braid (1) and its inverse, $\mathcal{V}$-Cat is a monoidal 2-category.

2.11. Remark.

Note that if in the definition of $\otimes^{n(1)}$ we replace $\otimes'^{(1)}$ with $\otimes^{(1)}$ then we have a multiplication equivalent to $\otimes^{[n-1](1)}$. Also note that

$$\mathcal{A} \otimes^{-n(1)} \mathcal{B} = (\mathcal{A}^{po^n} \otimes^{(1)} \mathcal{B}^{po^n})^{op^n}.$$

Notice that in the symmetric case the axioms of enriched categories for $\mathcal{A} \otimes^{(1)} \mathcal{B}$ and the existence of a coherent 2-natural transformation follow from the coherence of symmetric categories and the enriched axioms for $\mathcal{A}$ and $\mathcal{B}$. It remains to consider just why it is that $\mathcal{V}$-Cat is braided if and only if $\mathcal{V}$ is symmetric, and if so, then $\mathcal{V}$-Cat is symmetric as well. A braiding $c^{(1)}$ on $\mathcal{V}$-Cat is a 2-natural transformation so $c^{(1)}_{\mathcal{AB}}$ is a $\mathcal{V}$-functor $\mathcal{A} \otimes^{(1)} \mathcal{B} \to \mathcal{B} \otimes^{(1)} \mathcal{A}$. On objects $c^{(1)}_{\mathcal{AB}}((A, B)) = (B, A)$. Now to be precise we define $c^{(1)}$ to be based upon c to mean that

$$c^{(1)}_{\mathcal{AB}_{(A,B)(A',B')}} : (\mathcal{A} \otimes^{(1)} \mathcal{B})((A, B), (A', B')) \to (\mathcal{B} \otimes^{(1)} \mathcal{A})((B, A), (B', A'))$$

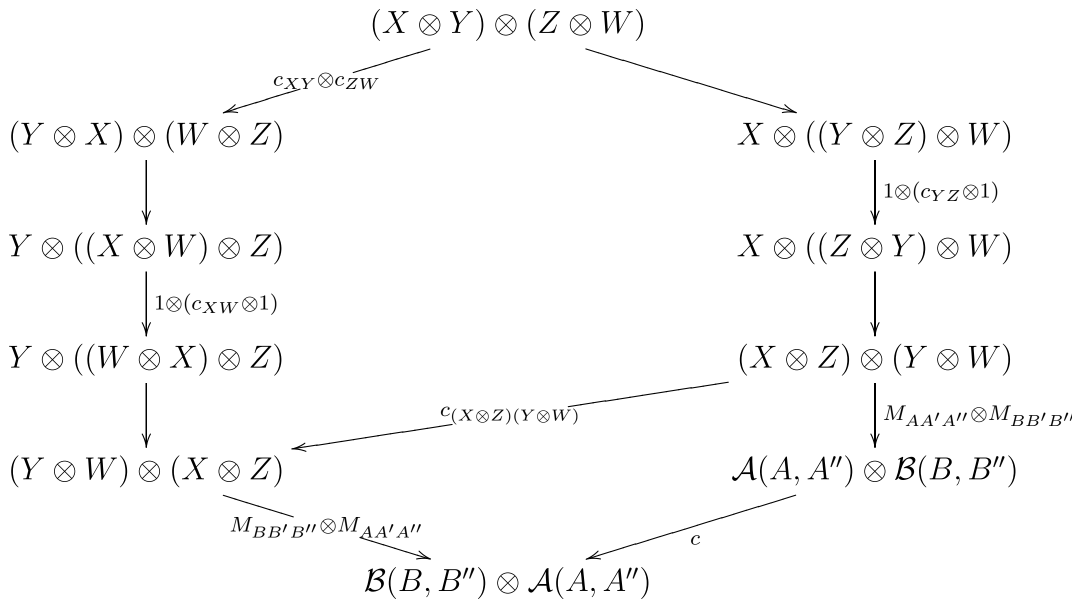
is exactly equal to

$$c_{\mathcal{A}(A,A')\mathcal{B}(B,B')} : \mathcal{A}(A, A') \otimes \mathcal{B}(B, B') \to \mathcal{B}(B, B') \otimes \mathcal{A}(A, A')$$

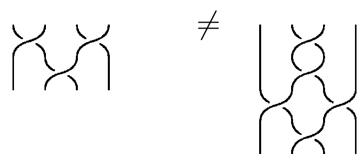
This potential braiding must be checked for $\mathcal{V}$-functoriality. Again the unit axioms are trivial and we consider the more interesting associativity of hom-object morphisms property. The following diagram must commute

$$\begin{array}{ccc}
(\mathcal{A} \otimes^{(1)} \mathcal{B})((A', B'), (A'', B'')) \otimes (\mathcal{A} \otimes^{(1)} \mathcal{B})((A, B), (A', B')) & \xrightarrow{M} & (\mathcal{A} \otimes^{(1)} \mathcal{B})((A, B), (A'', B'')) \\
\downarrow{\scriptstyle c^{(1)} \otimes c^{(1)}} & & \downarrow{\scriptstyle c^{(1)}} \\
(\mathcal{B} \otimes^{(1)} \mathcal{A})((B', A'), (B'', A'')) \otimes (\mathcal{B} \otimes^{(1)} \mathcal{A})((B, A), (B', A')) & \xrightarrow{M} & (\mathcal{B} \otimes^{(1)} \mathcal{A})((B, A), (B'', A''))
\end{array}$$

Let $X = \mathcal{A}(A', A'')$, $Y = \mathcal{B}(B', B'')$, $Z = \mathcal{A}(A, A')$ and $W = \mathcal{B}(B, B')$ Then expanding the above diagram using the composition defined as above (denoting various composites of α by unlabeled arrows) we have



The bottom quadrilateral commutes by naturality of c. The top region must then commute for the diagram to commute, but the left and right legs have the following underlying braids



Thus as noted in [Joyal and Street, 1993] neither braid (1) nor its inverse can in general give a monoidal structure with a braiding based on the original braiding. In fact, it is easy to show more.

2.12. Theorem. *Composition morphisms for product enriched categories with any underlying braid x will fail to produce a braiding in $\mathcal{V}$-Cat based upon the braiding in $\mathcal{V}$.*

Proof: Notice that in the above braid inequality each side of the inequality consists of the braid that underlies the definition of the composition morphism, in this case $b_{(1)}$, and an additional braid that underlies the segment of the preceding diagram that corresponds to a composite of $c^{(1)}$. In terms of braid generators the left side of the braid inequality begins with $\sigma_1\sigma_3$ corresponding to $c_{XY} \otimes c_{ZW}$ and the right side of the braid inequality ends with $\sigma_2\sigma_1\sigma_3\sigma_2$ corresponding to $c_{(X \otimes Z)(Y \otimes W)}$. Since the same braid x must end the left side as begins the right side, then for the diagram to commute we require $x\sigma_1\sigma_3 = \sigma_2\sigma_1\sigma_3\sigma_2 x$. This implies $\sigma_1\sigma_3 = x^{-1}\sigma_2\sigma_1\sigma_3\sigma_2 x$, or that the braids $\sigma_1\sigma_3$ and $\sigma_2\sigma_1\sigma_3\sigma_2$ are conjugate. Conjugate braids have precisely the same link as their closures, but the closure of $\sigma_1\sigma_3$ is an unlinked pair of circles whereas the closure of $\sigma_2\sigma_1\sigma_3\sigma_2$ is the Hopf link.

2.13. Remark.

It is also interesting to note that the braid inequality above is the 180 degree rotation of the one which implies that $(\mathcal{A}\otimes^{(1)}\mathcal{B})^{op} \neq \mathcal{A}^{op}\otimes^{(1)}\mathcal{B}^{op}$. Thus the proof also implies that the latter inequality holds for product enriched categories with any braid x underlying their composition morphisms.

2.14. Remark.

It is quickly seen that if c is a symmetry then in the second half of the braid inequality the upper portion of the braid consists of c_{YZ} and $c_{ZY} = c_{YZ}^{-1}$ so in fact equality holds. In that case then the derived braiding $c^{(1)}$ is a symmetry simply due to the definition.

3. 2-fold Monoidal Categories

In this section we closely follow the authors of [Balteanu et.al, 2003] in defining a notion of iterated monoidal category. For those readers familiar with that source, note that we vary from their definition only by including associativity up to natural coherent isomorphisms. This includes changing the basic picture from monoids to something that is a monoid only up to a monoidal natural transformation.

3.1. Definition. *A monoidal functor $(F,\eta) : \mathcal{C} \to \mathcal{D}$ between monoidal categories consists of a functor F such that $F(I) = I$ together with a natural transformation*

$$\eta_{AB} : F(A) \otimes F(B) \to F(A \otimes B),$$

which satisfies the following conditions

1. *Internal Associativity: The following diagram commutes*

$$\begin{array}{ccc}
(F(A)\otimes F(B))\otimes F(C) & \xrightarrow{\eta_{AB}\otimes 1_{F(C)}} & F(A\otimes B)\otimes F(C) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \eta_{(A\otimes B)C}} \\
F(A)\otimes(F(B)\otimes F(C)) & & F((A\otimes B)\otimes C) \\
\downarrow{\scriptstyle 1_{F(A)}\otimes\eta_{BC}} & & \downarrow{\scriptstyle F\alpha} \\
F(A)\otimes F(B\otimes C) & \xrightarrow{\eta_{A(B\otimes C)}} & F(A\otimes(B\otimes C))
\end{array}$$

2. *Internal Unit Conditions: $\eta_{AI} = \eta_{IA} = 1_{F(A)}$.*

Given two monoidal functors $(F,\eta) : \mathcal{C} \to \mathcal{D}$ and $(G,\zeta) : \mathcal{D} \to \mathcal{E}$, we define their composite to be the monoidal functor $(GF,\xi) : \mathcal{C} \to \mathcal{E}$, where ξ denotes the composite

$$GF(A)\otimes GF(B) \xrightarrow{\zeta_{F(A)F(B)}} G\big(F(A)\otimes F(B)\big) \xrightarrow{G(\eta_{AB})} GF(A\otimes B).$$

It is easy to verify that ξ satisfies the internal associativity condition above by subdividing the necessary commuting diagram into two regions that commute by the axioms for η and ζ respectively and two that commute due to their naturality. **MonCat** is the monoidal category of monoidal categories and monoidal functors, with the usual Cartesian product as in **Cat**.

A *monoidal natural transformation* $\theta : (F, \eta) \to (G, \zeta) : \mathcal{D} \to \mathcal{E}$ is a natural transformation $\theta : F \to G$ between the underlying ordinary functors of F and G such that the following diagram commutes

$$\begin{array}{ccc} F(A) \otimes F(B) & \xrightarrow{\eta} & F(A \otimes B) \\ \downarrow{\scriptstyle \theta_A \otimes \theta_B} & & \downarrow{\scriptstyle \theta_{A \otimes B}} \\ G(A) \otimes G(B) & \xrightarrow{\zeta} & G(A \otimes B) \end{array}$$

3.2. Definition. *For our purposes a 2-fold monoidal category is a tensor object in* **MonCat**. *This means that we are given a monoidal category* $(\mathcal{V}, \otimes_1, \alpha^1, I)$ *and a monoidal functor* $(\otimes_2, \eta) : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ *which satisfies*

1. *External Associativity: the following diagram describes a monoidal natural transformation* α^2 *in* **MonCat**.

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{V} \times \mathcal{V} & \xrightarrow{(\otimes_2, \eta) \times 1_{\mathcal{V}}} & \mathcal{V} \times \mathcal{V} \\ {\scriptstyle 1_{\mathcal{V}} \times (\otimes_2, \eta)} \downarrow & \swarrow\!\!\swarrow {\scriptstyle \alpha^2} & \downarrow {\scriptstyle (\otimes_2, \eta)} \\ \mathcal{V} \times \mathcal{V} & \xrightarrow[(\otimes_2, \eta)]{} & \mathcal{V} \end{array}$$

2. *External Unit Conditions: the following diagram commutes in* **MonCat**

$$\begin{array}{ccccc} \mathcal{V} \times I & \xrightarrow{\subseteq} & \mathcal{V} \times \mathcal{V} & \xleftarrow{\supseteq} & I \times \mathcal{V} \\ & {\scriptstyle \cong} \searrow & \downarrow {\scriptstyle (\otimes_2, \eta)} & \swarrow {\scriptstyle \cong} & \\ & & \mathcal{V} & & \end{array}$$

3. *Coherence: The underlying natural transformation* α^2 *satisfies the usual coherence pentagon.*

Explicitly this means that we are given a second associative binary operation $\otimes_2 : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, for which I is also a two-sided unit. We are also given a natural transformation

$$\eta_{ABCD} : (A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D).$$

The internal unit conditions give $\eta_{ABII} = \eta_{IIAB} = 1_{A\otimes_2 B}$, while the external unit conditions give $\eta_{AIBI} = \eta_{IAIB} = 1_{A\otimes_1 B}$. The internal associativity condition gives the commutative diagram

$$\begin{array}{ccc}
((U \otimes_2 V) \otimes_1 (W \otimes_2 X)) \otimes_1 (Y \otimes_2 Z) & \xrightarrow{\eta_{UVWX}\otimes_1 1_{Y\otimes_2 Z}} & \big((U \otimes_1 W) \otimes_2 (V \otimes_1 X)\big) \otimes_1 (Y \otimes_2 Z) \\
\downarrow{\scriptstyle \alpha^1} & & \downarrow{\scriptstyle \eta_{(U\otimes_1 W)(V\otimes_1 X)YZ}} \\
(U \otimes_2 V) \otimes_1 ((W \otimes_2 X) \otimes_1 (Y \otimes_2 Z)) & & ((U \otimes_1 W) \otimes_1 Y) \otimes_2 ((V \otimes_1 X) \otimes_1 Z) \\
\downarrow{\scriptstyle 1_{U\otimes_2 V}\otimes_1 \eta_{WXYZ}} & & \downarrow{\scriptstyle \alpha^1 \otimes_2 \alpha^1} \\
(U \otimes_2 V) \otimes_1 \big((W \otimes_1 Y) \otimes_2 (X \otimes_1 Z)\big) & \xrightarrow{\eta_{UV(W\otimes_1 Y)(X\otimes_1 Z)}} & (U \otimes_1 (W \otimes_1 Y)) \otimes_2 (V \otimes_1 (X \otimes_1 Z))
\end{array}$$

The external associativity condition gives the commutative diagram

$$\begin{array}{ccc}
((U \otimes_2 V) \otimes_2 W) \otimes_1 ((X \otimes_2 Y) \otimes_2 Z) & \xrightarrow{\eta_{(U\otimes_2 V)W(X\otimes_2 Y)Z}} & \big((U \otimes_2 V) \otimes_1 (X \otimes_2 Y)\big) \otimes_2 (W \otimes_1 Z) \\
\downarrow{\scriptstyle \alpha^2 \otimes_1 \alpha^2} & & \downarrow{\scriptstyle \eta_{UVXY}\otimes_2 1_{W\otimes_1 Z}} \\
(U \otimes_2 (V \otimes_2 W)) \otimes_1 (X \otimes_2 (Y \otimes_2 Z)) & & ((U \otimes_1 X) \otimes_2 (V \otimes_1 Y)) \otimes_2 (W \otimes_1 Z) \\
\downarrow{\scriptstyle \eta_{U(V\otimes_2 W)X(Y\otimes_2 Z)}} & & \downarrow{\scriptstyle \alpha^2} \\
(U \otimes_1 X) \otimes_2 \big((V \otimes_2 W) \otimes_1 (Y \otimes_2 Z)\big) & \xrightarrow{1_{U\otimes_1 X}\otimes_2 \eta_{VWYZ}} & (U \otimes_1 X) \otimes_2 ((V \otimes_1 Y) \otimes_2 (W \otimes_1 Z))
\end{array}$$

Notice that these are precisely the diagrams from Section 2 that were needed to commute respectively in the questions of the associativity of the composition and the functoriality of the associator.

The authors of [Balteanu et.al, 2003] remark that we have natural transformations

$$\eta_{AIIB} : A \otimes_1 B \to A \otimes_2 B \qquad \text{and} \qquad \eta_{IABI} : A \otimes_1 B \to B \otimes_2 A.$$

If they had insisted a 2-fold monoidal category be a tensor object in the category of monoidal categories and *strictly monoidal* functors, this would be equivalent to requiring that $\eta = 1$. In view of the above, they note that this would imply $A\otimes_1 B = A\otimes_2 B = B\otimes_1 A$ and similarly for morphisms.

Joyal and Street [Joyal and Street, 1993] considered a similar concept to Balteanu, Fiedorowicz, Schwänzl and Vogt's idea of 2-fold monoidal category. The former pair required the natural transformation η_{ABCD} to be an isomorphism and showed that the resulting category is naturally equivalent to a braided monoidal category. As explained in [Balteanu et.al, 2003], given such a category one obtains an equivalent braided monoidal category by discarding one of the two operations, say $\otimes_2$, and defining the commutativity isomorphism for the remaining operation $\otimes_1$ to be the composite

$$A \otimes_1 B \xrightarrow{\eta_{IABI}} B \otimes_2 A \xrightarrow{\eta^{-1}_{BIIA}} B \otimes_1 A.$$

Here we choose here to pass to an equivalent strictly associative category for simplicity. In [Balteanu et.al, 2003] it is shown that a 2-fold monoidal category with $\otimes_1 = \otimes_2$, η an isomorphism and

$$\eta_{AIBC} = \eta_{ABIC} = 1_{A\otimes B\otimes C}$$

is a braided monoidal category with the braiding $c_{BC} = \eta_{IBCI}$.

Also note that for $\mathcal{V}$ braided the interchange given by $\eta_{ABCD} = 1_A \otimes c_{BC} \otimes 1_D$ gives a 2-fold monoidal category where $\otimes_1 = \otimes_2$.

In this setting we ask whether, based on a braiding, there are alternate 2-fold monoidal structures on $\mathcal{V}$, with $\otimes_1 = \otimes_2$. Of course we already know the answer, since the required unit and associativity axioms are precisely satisfied by those compositions of the braiding such as

$$\eta_{ABCD} = (c^{-1}_{CA} \otimes c^{-1}_{DB}) \circ (1_C \otimes c_{DA} \otimes 1_B) \circ (c_{(A\otimes B)(C\otimes D)}),$$

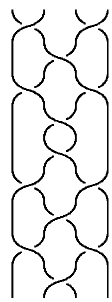
with underlying braid $b_{(5)}$. Thus the same family of braids as was discovered to underlie the composition in nth left opposites of products of nth right opposites of enriched categories can be used to define valid interchangers to make $\mathcal{V}$ into a 2-fold monoidal category. The first gain realized from considering 2-fold monoidal structures is that since they are simpler they provide opportunity to uncover obstructions in certain classes of braids that render the corresponding compositions of the braiding on $\mathcal{V}$ ineligible to be an interchanger. Of course we are restricting our inquiry to those elements of B_4 which are potential candidates for underlying an interchanger made up of a composition of instances of the braiding. In the current section an associative braid b is one which obeys the internal and external associativity axioms. In the terms of the last section $Lb = Rb$ and $L'b = R'b$.

The general scheme is to find extra conditions on the interchanger η which together with the unit conditions and the associativity conditions will force the underlying braid to have easily checked characteristics. Then we can find families of braids which cannot underlie an interchanger due to non-associativity since they obey the first set of conditions but lack the predicted characteristics. We present several examples here, but the full theory is incomplete until it decides the question of associativity for every candidate braid. A major future project is to complete this effort in part by exhausting the present approach and if necessary by finding obstructions in braid invariants.

3.3. Theorem. *Given an interchange candidate braid b with the property that deleting either the 2nd or 3rd strand gives the identity braid on three strands, then b is associative if and only if $b = \sigma_2$, the second generator of B_4, or its inverse.*

Proof: This follows the logic of [Balteanu et.al, 2003]. Letting η_{ABCD} be the interchanger based on the braiding of $\mathcal{V}$ with underlying braid b, note that deleting strands is equivalent to replacing the corresponding object in the product $A \otimes B \otimes C \otimes D$ with the identity I. Now let $V = W = I$ in the internal associativity diagram to see that due to the hypotheses on b we have that $\eta_{UXYZ} = 1_U \otimes \eta_{IXYZ}$. Then let X=Y=I in the internal associativity diagram to see that $\eta_{UVWZ} = \eta_{UVWI} \otimes 1_Z$. Together these two facts imply that $\eta_{ABCD} = 1_A \otimes \eta_{IBCI} \otimes 1_D$. Then if we take $U = Z = W = 0$ in the internal associativity law we get the first axiom of a braided category for $c'_{BC} = \eta_{IBCI}$, and letting $U = Z = X = 0$ in the internal associativity diagram gives the other one. This then implies that either $c' = c$ or $c' = c^{-1}$, since no other combinations of c give a braiding. Therefore $\eta_{ABCD} = 1_A \otimes c^{\pm 1}_{BC} \otimes 1_D$ which has the underlying braid $\sigma_2^{\pm 1}$. The converse is also clear from this discussion, since all the implications can be reversed.

This sort of obstruction can rule out candidate braids such as the braid $b_{(4)}$ in the last section. More generally it also rules out all but one element each of the left and right $\sigma_2^{\pm(2n-1)}$-cosets of the Brunnian braids in B_4, where the Brunnian braids are those pure braids where any strand deletion gives the identity braid. Even more broadly this obstruction rules out braids such as:



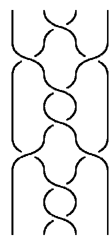
Notice that the longer associative braids found in the last section give examples of interchangers that do not fit the conditions of this theorem. They also serve as examples of interchangers η such that η_{IBCI} is not a braiding. It is nice to check however that they do give a braiding via $c'_{AB} = \eta^{-1}_{BIIA} \circ \eta_{IABI}$ as predicted by Joyal and Street. The latter condition also serves as a source of obstructions on its own. According to their theorem, any associative braid will have the property that dropping the outer two strands will give a two strand braid with one more crossing of the same handedness than the two strand braid achieved by dropping the inner two strands. Indeed this condition rules out some of the same braids just mentioned, namely the Brunnian cosets of higher powers of σ_2 in B_4.

The next sort of obstruction is found by slightly weakening the extra conditions. This will allow us to rule out a larger, different class of candidates, but they will be a little bit harder to recognize.

3.4. Theorem. *Let b be an interchange candidate braid with the property that deleting either the first or the fourth strand results in a 3-strand braid that is just a power of the braid generator on what were the middle two strands: $\sigma_i^{\pm n}$; $i = 2$ or $i = 1$ respective of whether the first or fourth strand was deleted. Then b is associative implies that $n = 1$.*

Proof: The strand deletion conditions on the underlying braid b of η are equivalent to assuming that $\eta_{IBCD} = \eta_{IBCI} \otimes 1_D$ and that $\eta_{ABCI} = 1_A \otimes \eta_{IBCI}$. Of course the power of the generator being ± 1 is equivalent to saying that η_{IBCI} is the braiding c or its inverse. Hence we need only show that the assumptions imply that η_{IBCI} is a braiding. This is seen immediately upon letting $U = Z = W = 0$ in the internal associativity axiom to get the first axiom of a braiding and letting $U = Z = X = 0$ to get the other one.

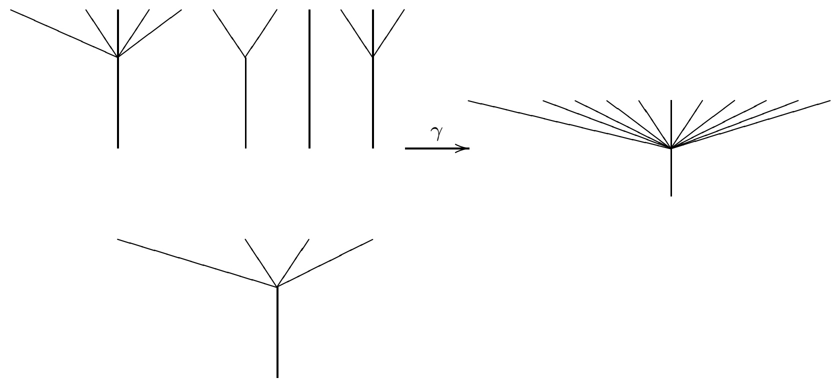
This theorem can be applied as a source of obstructions in two ways. First it can directly rule out candidates which satisfy the Joyal and Street condition that $c_{AB} = \eta^{-1}_{BIIA} \circ \eta_{IABI}$ and the first or last strand deletion condition given here, but which fail to give a single crossing braid upon that removal. The simplest example is this braid:



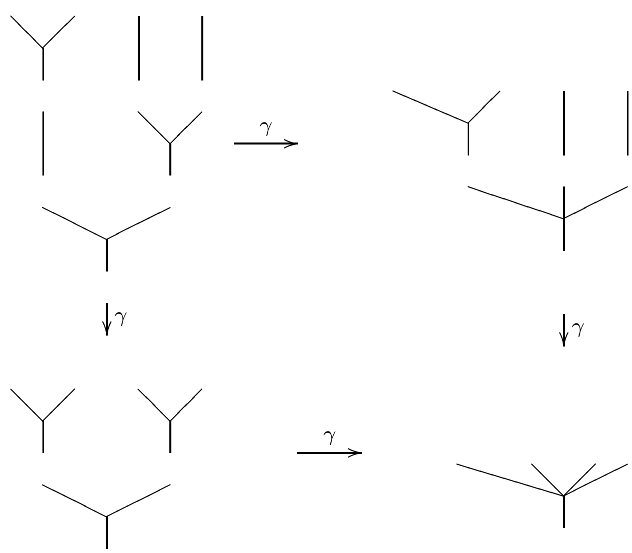
Secondly one can use this result in the following form: The proof also implies that a candidate braid which yields a single crossing after deletion of the first and fourth strands, if associative, must then obey the condition that deleting the first or last strand frees the other of those two from any crossings. This rules out for example braids $b_{(2)}$ and $b_{(3)}$ from the Examples of the last section.

4. 2-fold operads

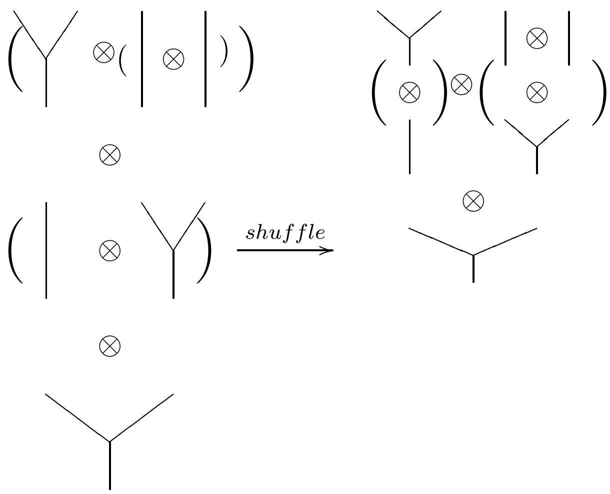
The middle exchange or interchanger shows up in quite a few mathematical settings. Whenever it is an isomorphism it leads to the Eckmann-Hilton principle that forces symmetry in dimensions higher than 2. Higher homotopy groups commute and a 3-fold monoidal category with isomorphism interchangers is symmetric. The braided and symmetric versions of an algebraic construction are often cited, but the interchanger and its associativity are actually more fundamental. So far herein we have found families of interchangers based on a braiding that can define either a 2-fold monoidal structure on a category or a monoidal structure on a 2-category, as well as families of obstructions that rule out certain alternate candidate interchangers. Another common use of a braiding is to define a monoidal structure on a category of collections, as in the theory of operads. The two principle components of an operad are a collection, historically a sequence, of objects in a monoidal category and a family of composition maps. Operads are often described as paramaterizations of n-ary operations. Peter May's original definition of operad in a symmetric (or braided) monoidal category [May, 1972] has a composition γ that takes the tensor product of the nth object (n-ary operation) and n others (of various arity) to a resultant that sums the arities of those others. The nth object or n-ary operation is often pictured as a tree with n leaves, and the composition appears like this:



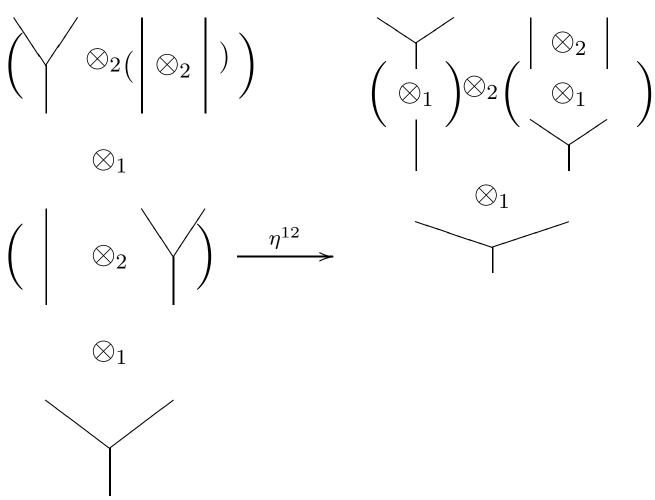
By requiring this composition to be associative we mean that it obeys this sort of pictured commuting diagram:



In the above pictures the tensor products are shown just by juxtaposition, but now we would like to think about the products more explicitly. If the monoidal category is not strict, then there is actually required another leg of the diagram, where the tensoring is reconfigured so that the composition can operate in an alternate order. Here is how that rearranging looks in a symmetric (braided) category, where the shuffling is accomplished by use of the symmetry (braiding):



We now foreshadow our definition of operads in an iterated monoidal category with the same picture as above but using two tensor products, $\otimes_1$ and $\otimes_2$. It now becomes clear that the true nature of the shuffle is in fact that of an interchange transformation. Indeed one point to be taken is that the braided case should actually be viewed as a 2-fold monoidal category with $\otimes_1 = \otimes_2$ and η given in terms of the braiding. In what follows for clarity we use η^{12} whenever there are two products in discussion.



To see this just focus on the actual domain and range of η^{12} which are the upper two levels of trees in the pictures, with the tensor product $(| \otimes_2 |)$ considered as a single object.

Now we are ready to give the technical definitions. We begin with the definition of 2-fold operad in a 2-fold monoidal category, as in the above picture, and then show how it generalizes the case of operad in a braided category.

Let $\mathcal{V}$ be an 2-fold monoidal category as defined in Section 2.

4.1. Definition. *A 2-fold operad $\mathcal{C}$ in $\mathcal{V}$ consists of objects $\mathcal{C}(j)$, $j \geq 0$, a unit map $\mathcal{J} : I \to \mathcal{C}(1)$, and composition maps in $\mathcal{V}$*

$$\gamma^{12} : \mathcal{C}(k) \otimes_1 (\mathcal{C}(j_1) \otimes_2 \ldots \otimes_2 \mathcal{C}(j_k)) \to \mathcal{C}(j)$$

for $k \geq 1$, $j_s \geq 0$ for $s = 1 \ldots k$ and $\sum_{n=1}^{k} j_n = j$. The composition maps obey the following axioms

1. *Associativity: The following diagram is required to commute for all $k \geq 1$, $j_s \geq 0$ and $i_t \geq 0$, and where $\sum_{s=1}^{k} j_s = j$ and $\sum_{t=1}^{j} i_t = i$. Let $g_s = \sum_{u=1}^{s} j_u$ and let $h_s = \sum_{u=1+g_{s-1}}^{g_s} i_u$.*

   The η^{12} labelling the leftmost arrow actually stands for a variety of equivalent maps which factor into instances of the 12 interchange.

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2\mathcal{C}(j_s) \right) \otimes_1 \left(\bigotimes_{t=1}^{j} {}_2\mathcal{C}(i_t) \right) & \xrightarrow{\gamma^{12} \otimes_1 \mathrm{id}} & \mathcal{C}(j) \otimes_1 \left(\bigotimes_{t=1}^{j} {}_2\mathcal{C}(i_t) \right) \\
 & & \downarrow \gamma^{12} \\
\downarrow \mathrm{id} \otimes_1 \eta^{12} & & \mathcal{C}(i) \\
 & & \uparrow \gamma^{12} \\
\mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2\mathcal{C}(j_s) \otimes_1 \left(\bigotimes_{u=1}^{j_s} {}_2\mathcal{C}(i_{u+g_{s-1}}) \right) \right) & \xrightarrow[\mathrm{id} \otimes_1 (\otimes_2^k \gamma^{12})]{} & \mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2\mathcal{C}(h_s) \right)
\end{array}$$

2. *Respect of units is required just as in the symmetric case. The following unit diagrams commute.*

$$\begin{array}{ccc}
\mathcal{C}(k) \otimes_1 (\otimes_2^k I) & = & \mathcal{C}(k) \\
\downarrow 1 \otimes_1 (\otimes_2^k \mathcal{J}) & \nearrow \gamma^{12} & \\
\mathcal{C}(k) \otimes_1 (\otimes_2^k \mathcal{C}(1)) & &
\end{array}
\qquad
\begin{array}{ccc}
I \otimes_1 \mathcal{C}(k) & = & \mathcal{C}(k) \\
\downarrow \mathcal{J} \otimes_1 1 & \nearrow \gamma^{12} & \\
\mathcal{C}(1) \otimes_1 \mathcal{C}(k) & &
\end{array}$$

Note that operads in a braided monoidal category are examples of 2-fold operads. This is true based on the arguments of Joyal and Street [Joyal and Street, 1993], who showed that braided categories arise as 2-fold monoidal categories where the interchanges are isomorphisms. Also note that given such a perspective on a braided category, the two products are equivalent and the use of the braiding to shuffle in the operad associativity requirement can be rewritten as the use of the interchange.

Operads in a symmetric (braided) monoidal category with coproducts are often efficiently defined as the monoids of a category of collections. For a braided category $\mathcal{V}$ the objects of $Col(\mathcal{V})$ are functors from the category of natural numbers to $\mathcal{V}$. In other words the data for a collection $\mathcal{C}$ is a sequence of objects $\mathcal{C}(j)$ just as for an operad. Morphisms in $Col(\mathcal{V})$ are natural transformations. The tensor product in $Col(\mathcal{V})$ is given by

$$(\mathcal{B} \otimes \mathcal{C})(j) = \coprod_{k\geq 0, j_1+\ldots+j_k=j} \mathcal{B}(k) \otimes (\mathcal{C}(j_1) \otimes \ldots \otimes \mathcal{C}(j_k))$$

where $j_i \geq 0$. This product is associative by use of the symmetry or braiding. The unit is the collection $(\emptyset, I, \emptyset, \ldots)$ where $\emptyset$ is an initial object in $\mathcal{V}$. Here we can observe how the interchange transformations generalize braiding. For $\mathcal{V}$ a 2-fold monoidal category define the objects and morphisms of $Col(\mathcal{V})$ in precisely the same way, but define the product to be

$$(\mathcal{B}\otimes^{12}\mathcal{C})(j) = \coprod_{k\geq 0, j_1+\ldots+j_k=j} \mathcal{B}(k) \otimes_1 (\mathcal{C}(j_1) \otimes_2 \ldots \otimes_2 \mathcal{C}(j_k))$$

Associativity is seen by inspection of the two products $(\mathcal{B}\otimes^{12}\mathcal{C})\otimes^{12}\mathcal{D}$ and $\mathcal{B}\otimes^{12}(\mathcal{C}\otimes^{12}\mathcal{D})$.

In the braided case mentioned above, the two coproducts in question are seen to be composed of the same terms up to a braiding between them. Here the terms of the two coproducts are related by instances of the interchange transformation η^{12} from the term in $((\mathcal{B}\otimes^{12}\mathcal{C})\otimes^{12}\mathcal{D})(j)$ to the corresponding term in $(\mathcal{B}\otimes^{12}(\mathcal{C}\otimes^{12}\mathcal{D}))(j)$. For example upon expansion of the two three-fold products we see that in the coproduct which is $((\mathcal{B}\otimes^{12}\mathcal{C})\otimes^{12}\mathcal{D})(2)$ we have the term

$$\mathcal{B}(2) \otimes_1 (\mathcal{C}(1) \otimes_2 \mathcal{C}(1)) \otimes_1 (\mathcal{D}(1) \otimes_2 \mathcal{D}(1))$$

while in $(\mathcal{B}\otimes^{12}(\mathcal{C}\otimes^{12}\mathcal{D}))(2)$ we have the term

$$\mathcal{B}(2) \otimes_1 (\mathcal{C}(1) \otimes_1 \mathcal{D}(1)) \otimes_2 (\mathcal{C}(1) \otimes_1 \mathcal{D}(1)).$$

The coherence theorem of iterated monoidal categories in [Balteanu et.al, 2003] guarantees the commutativity of the pentagon equation. Now we have a condensed way of defining 2-fold operads.

4.2. Theorem. *2-fold operads in 2-fold monoidal $\mathcal{V}$ are monoids in $Col(\mathcal{V})$.*

Proof: A monoid in $Col(\mathcal{V})$ is an object $\mathcal{C}$ in $Col(\mathcal{V})$ with multiplication and unit morphisms. Since morphisms of $Col(\mathcal{V})$ are natural transformations the multiplication and unit consist of families of maps in $\mathcal{V}$ indexed by the natural numbers, with source and target exactly as required for operad composition and unit. The operad axioms are equivalent to the associativity and unit requirements of monoids.

Now the problem of describing the various sorts of operads in a braided monoidal category becomes more clear, as a special case. Here again we let $\otimes = \otimes_1 = \otimes_2$. The families of 2-fold structures based on associative braids give rise to families of monoidal structures on the category of collections, and thus to families of operad structures.

In the operad picture the underlying braid of an operad structure only becomes important when we inspect the various ways of composing a product such as $\mathcal{C}(2) \otimes (\mathcal{C}(1) \otimes \mathcal{C}(1)) \otimes (\mathcal{C}(1) \otimes \mathcal{C}(1)) \otimes (\mathcal{C}(1) \otimes \mathcal{C}(1))$. For this composition to be well defined we require the internal associativity of the interchange that is used to rearrange the terms. As soon as we consider composing a product of similar height, i.e. of four levels of trees in the heuristic diagram, but with a base term $\mathcal{C}(n)$ with $n \geq 3$, such as: $\mathcal{C}(3) \otimes (\mathcal{C}(1) \otimes \mathcal{C}(1) \otimes \mathcal{C}(1)) \otimes (\mathcal{C}(1) \otimes \mathcal{C}(1) \otimes \mathcal{C}(1)) \otimes (\mathcal{C}(1) \otimes \mathcal{C}(1) \otimes \mathcal{C}(1))$, then we see that the external associativity of η is also required.

Thus the same theorems proven above for associative and nonassociative families of braids apply here as well, in deciding whether a certain braid based shuffling of the terms in an operad product is allowable. The point is that not all shuffles using a braiding make sense, and the viewpoint of the 2-fold monoidal structure is precisely what is needed to see which shuffles do make sense. By seeing various shuffles as being interchanges on a fourfold product rather than braidings on a simple binary product, we are able to describe infinite families of distinct compositions of the braiding leading to well defined operad structure. These are the same families based on the left and right opposites of enriched categories as are detailed in Section 2. We are also able to eliminate many other potential candidates for operad structure via precisely the obstructions studied in Section 3. In summary, structures based on a braiding are ill-defined unless a 2-fold monoidal structure is chosen. Often in the literature the default is understood to be the simplest such structure where $\eta_{ABCD} = 1_A \otimes c_{BC} \otimes 1_D$, but to be careful this choice should be made explicit. For example operads in a braided monoidal category are not well defined, whereas operads in a 2-fold monoidal category based upon that braiding are. The monoidal structure on the category of enriched categories over a braided monoidal category is not really well defined, but if a 2-fold monoidal structure is chosen then it is well defined. The general definition of a category enriched over a 2-fold monoidal category is not detailed herein, but can be found in [Forcey, 2004]. Future work should investigate the effects of the choice of 2-fold monoidal structure on the super-structure based upon it. For instance we would like to know what effect our choice of associative braid has on the category of operads in $\mathcal{V}$, especially when those operads come with an action of the braid groups.

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